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Numerical treatment of Differential Equations II, 2D1255, 2005

Problem set 7

Hand in a solution no later then May 31.

A very much simplified model for the electron density $N(x, E, t)$ in the vicinity of a black hole is

$$N_t = (\kappa_1 N_x)x + (\kappa_2(x) N_E)_E + Q(x, E).$$

Here $Q(x, E)$ models the electron production due to dark matter annihilation. We want to compute a steady state $\tilde{N}(x, E)$ and determine

$$\int_{x_0}^{x_1} \tilde{N}(x, E) dx$$

for different values of E. It is also interesting to know the approximate time it takes to reach steady state.

When measuring time t in seconds, the space coordinate x in $10^3 pc = kpc$ and the energy E in $10^9 eV = GeV$ the coefficients take numerical values

$$\kappa_1 = 1.48 \cdot 10^{18}, \quad \kappa_2(x) = 4.04 \cdot 10^{10} x^{-0.5}.$$

You may omit κ_2 dependence on x in this study. We want to consider the domain

$$x_0 = 1 < x < 100 = x_1, \quad 1 < E < 10,$$

with boundary conditions

$$N_E(x, 1, t) = N_E(x, 10, t) = 0, \quad N(x_0, E, t) = N(x_1, E, t) = 0.$$

Choose one of the following two source functions

$$Q_1(x, E) = e^{-(x-10)^2 - 10(E-5)^2}, \quad Q_2(x, E) = e^{-(x-10)^2} \delta(E - 10).$$

Note that in the second case we can replace the source function by a boundary condition at $E = 10$:

$$N_E(x, 10, t) = e^{-(x-10)^2}.$$

1. The fast technique can be used if the coefficients are independent of one of the directions. Write a program solving for the simple case

   $$\kappa_1 = 1, \quad \kappa_2(x) = 1.$$

   Plot the steady state.

2. Change the values of the coefficients. What happens to the solution? Can you solve for the values like the real values?

3. **Scaling** Introduce
$$t = T\tilde{t}, \quad x = X\tilde{x}, \quad E = E_0\tilde{E}, .$$
Choose values of the scaling parameters such that the diffusion coeffisients of the scaled problem are approximatly of order 1. Formulate the rescaled problem. Dont forget the boundary conditions.

4. Now the difficulty is the size of the domain. For both source terms one expects the solution to be essentially nonzero in only part of the domain. Which? To compute a numerical solution use one of the following approaches
   - Solve on a much smaller domain. Check the accuracy by moving the boundary and measuring how much the solution changes.
   - Use a very stretched grid.

5. Plot the integral as a function of E, and for the maximum value, plot the integral vaalue as a function of time.