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Numerical treatment of Differential Equations II, 2D1255, 2005

Homework problem 2

Hand in a report no later then Febuary 20.

Linear and nonlinear hyperbolic systems

In this exercise we shall investigate the relation between a non-linear problem and the corresponding linearized system. In particular we will see how well linear analysis predicts the behaviour of the nonlinear problem.

Consider the system

$$h_t + (vh)_x = 0,$$
$$v_t + (v^2 + h)_x = 0.$$

for $0 \leq x \leq 4$, $t \geq 0$. This is the so called shallow water equations. Initial and boundary conditions are

$$h(x,0) = 4 + \varepsilon e^{-\frac{(x-2)^2}{0.3}}, \quad v(x,0) = 0,$$

and

$$v(0,t) = v(4,t) = 0.$$

1. **Numerical Solution** To begin with let $\varepsilon = 0.1$ and solve numerically using the Lax-Friedrichs method. Use ghost cells at the boundaries. Prescribe values there by the technique of decomposing into characteristic variables or by the procedure described in chapter 7 for solid walls. Choose Δx and Δt after making numerical experiments with different discretization parameters. Use $\Delta t/\Delta x$ as large as possible, without violating stability. Make plots showing wave propagation and reflections at the boundaries. Compute at least until waves have been reflected at both boundaries and crossed each other.

   Run the program again with larger values of ε ($\varepsilon = 1, 2, ..$). How does the solution change?

2. **Linearization** Choose a relevant constant state and derive the linearized problem at that state. Don't forget initial and boundary conditions. Show that the linear problem is hyperbolic.

3. **Analysis of Linear Problem** The linear problem can be solved analytically. Determine the solution of the linear problem at for instance $t = 0.5$. Discuss how information propagats, if the boundary conditions cause reflections and when reflected waves will appear. Compare with the numerical results for the non-linear case.

4. **Non-Reflecting boundary condition** Derive boundary conditions for the linear problem that do not cause reflections. Formulate corresponding conditions for the non-linear case. Implement the conditions in your program using either the technique of characteristic variables or by simply extrapolating all variables at the boundary (as described in chapter 7). How well does the method work?