Examination paper Numerical Solution of DE, 2D1255/DN2255

9-14, Nov 12, 2007

Closed books examination. Read all the questions before starting work. Check carefully that your initially derived equations are correct. Ask if you are uncertain. Answers MUST be motivated. Paginate and write your name on EVERY page handed in.

A total of 19 out of max 39 points guarantees a "pass". The results will be e-mailed to participants by Dec. 1, 2007. Papers are kept at the CSC Student Office for a year and then destroyed. Complaints to J.Oppelstrup by Dec. 20, 2007, after which the results are irrevocable. The next examination paper will be given in May, 2008

P1.(9)

Consider the initial-boundary value problem on the strip $0 \le x \le 1$, $t \ge 0$:

$$\mathbf{q}_t + \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\mathbf{q}_x = 0, \mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$v = 0 \text{ at } x = 0$$

$$u + \alpha v = 0 \text{ at } x = 1$$

$$u(x,0) = \exp(-500(x-0.5)^2), v(x,0) = au(x,0)$$

(α, a real; A short pulse centered at $x = 0.5$, details not important)

(3) a) Diagonalize the system and find the characteristic speeds and variables (usually called w1 and w2.)

(2) b) Show that the boundary conditions give a well-posed problem *except* for one value of α. Which? Why?

(4) c) Find two values of a to make the solution be a single traveling pulse (not two). Sketch the solution for these values of a, for w1 and w2 ***OR*** for u and v. **Hint** eigenvectors !

P2. (6)

(3) (a) Derive the upwind method for the advection equation $q_t + uq_x = 0$ from the following algorithm.

1) Reconstruct a piecewise constant function $\tilde{q}^n(x)$ from cell averages Q_j^n

2) Solve $q_t + uq_x = 0, q(x,t^n) = \tilde{q}^n(x), t > t^n$ until $t = t^n + \Delta t = t^{n+1}$

3) Average $q(x,t^{n+1})$ over grid cells to obtain new cell averages Q_j^{n+1}.

(3) (b) Discuss how the reconstruction step can be modified to obtain a high resolution method. You need to explain the use of "slope limiters".

P3. (6)

(3) (a) Linearize the adiabatic gas dynamics equations

$$\begin{cases} \rho_t + m_x = 0, m = \rho u \\ m_t + (mu + K\rho^\gamma)_x = 0 \end{cases}$$

where ρ = density, u = velocity, and constants $K > 0$, $\gamma > 1$, (cont'd) around $\rho = R$, $m = M$ (constants). What are the characteristic speeds?

(3) (b) Consider the linearized problem for $0 \le x \le 1$. Suggest boundary conditions at $x = 0$ and $x = 1$ that yield a mathematically well posed problem. If you did not do (a), use $\begin{cases} r_t + m_x = 0 \\ m_t + (c^2 - a^2)r_x + 2am_x = 0 \end{cases}$

Note: No need to compute eigenvectors. There are several cases, dependent on the Mach-number $M = \dfrac{|U|}{\sqrt{K\gamma R^{\gamma-1}}}$ or $\dfrac{|a|}{|c|}$. Explain!

P4. (8)

(2) (a) Define the meaning of $\mathbf{A}^+$, $\mathbf{A}^-$, and $|\mathbf{A}|$ as used in the Roe-scheme. $\mathbf{A}$ is a real square matrix.
Let the system of conservation laws be $\mathbf{q}_t + (\mathbf{f}(\mathbf{q}))_x = 0$.

(1) (b) Define the (…) of a time-stepping scheme in conservation form

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}(...)$$

(2) (c) The numerical flux $\mathbf{F}_{j-1/2}^n$ should satisfy a consistency condition … which?

(1) (d) The Roe scheme has numerical flux

$$\mathbf{F}_{j-1/2}^n = \frac{1}{2}\left(\mathbf{f}(\mathbf{Q}_{j-1}^n) + \mathbf{f}(\mathbf{Q}_j^n)\right) - \frac{1}{2}\left|\mathbf{A}_{j-1/2}^n\right|\left(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n\right)$$

Define the "flux Jacobian matrix" and its relation to $\mathbf{A}_{j-1/2}$.

(2) (e) Suppose all eigenvalues of $\mathbf{A}$ have equal magnitude, c. Then $|\mathbf{A}|$ becomes very simple. What?

P5. (8)

(2) (a) Define what is meant by *prolongation* and *restriction* in the Multi-grid method.

(2) (b) Give numerical examples for a fine grid with 6 cells and a coarse grid with 3 cells: Write the matrices associated with prolongation by linear interpolation, and the injection and "full weighting" restrictions.

(4) (c) The 1-D elliptic problem $u_{xx} = f(x), u(0) = u(1) = 0$ is discretized by central differences into $u_{j-1} - 2u_j + u_{j+1} = \Delta x^2 f(x_j), j = 1,...,n, u_0 = u_{n+1} = 0$ with solution U_j. The damped Jacobi iteration

$$u_j^{n+1} = (1-\alpha)u_j^n + \alpha(-\Delta x^2 f_j + u_{j-1}^n + u_{j+1}^n)/2$$
$$u_j^0 = 0,$$

is applied. Write the iteration for the error $w_j^n = U_j - u_j^n$. Use von Neumann analysis with ansatz $w_j^n = \hat{w}^n \cdot e^{ikj\Delta x} = \hat{w}^n \cdot e^{ij\theta}, \theta = k\Delta x$. What range of θ must we consider? What are the "high"- and "low"-frequency ranges? Suppose the initial errors w_j^0 are compsed only of high-frequency components. What choice of a will most rapidly decrease the errors?