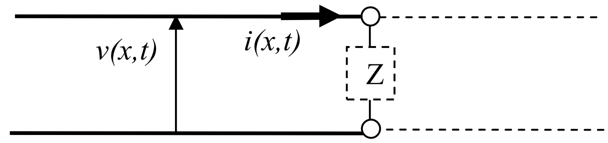
Lecture notes Feb 18, 2008

Non-Reflective Boundaries: termination of a transmission line

The telegrapher's equation

$$lI_t + v_x = 0$$
$$cv_t + I_x = 0$$



models an electric transmission line with capacitance c [F/m] and inductance l [H/m]. The voltage is v and the current I. It is a linear hyperbolic system

$$\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0, \mathbf{q} = \begin{pmatrix} I \\ v \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 & 1/l \\ 1/c & 0 \end{pmatrix}, c_0^2 = 1/lc, \text{impedance } Z_0 = \sqrt{l/c}$$

with wave speeds $\lambda+$, $\lambda- = +\text{-}c_0$, right eigenvectors $\mathbf{r}_+,\mathbf{r}_-$, characteristic variables w_+,w_- .

$$\mathbf{r}_+ = \begin{pmatrix} 1 \\ Z_0 \end{pmatrix}, \mathbf{r}_- = \begin{pmatrix} -1 \\ Z_0 \end{pmatrix}, \mathbf{q} = w_+\mathbf{r}_+ + w_-\mathbf{r}_-, w_+ = \frac{1}{2Z_0}(Z_0 I + v), w_- = \frac{1}{2Z_0}(-Z_0 I + v),$$

so the right- and left-running waves are

$$\mathbf{q}_R(x,t) = f(x - c_0 t)\cdot\begin{pmatrix} 1 \\ Z_0 \end{pmatrix}, \mathbf{q}_L(x,t) = g(x + c_0 t)\cdot\begin{pmatrix} -1 \\ Z_0 \end{pmatrix}$$

The line is terminated by an impedance Z, so $v = ZI$ at $x = L$. If we choose $Z = Z_0$, w_- is zero and there are no left-running waves: a perfectly non-reflective condition. In this case, infinite extension of the line to the right can be replaced by a boundary condition.

Implementation of condition at x_N

The ghost cell implementation, for a three-point scheme, so only one ghost cell: N+1.
w- is set to the incoming wave amplitude (0 here). w+ is computed from inside the computational domain by

a) characteristic extrapolation (Draw a sketch!)

$$w_{N+1}^{n+1} = (1 - \lambda^+ \frac{\Delta t}{\Delta x}) w_{N+1}^n + \lambda^+ \frac{\Delta t}{\Delta x} w_N^n$$

b) space extrapolation

$$w_{N+1}^{n+1} = w_N^{n+1} \text{ (zeroth order)}$$
$$w_{N+1}^{n+1} = 2w_N^{n+1} - w_{N-1}^{n+1} \text{ (first order)}$$

a) is equivalent to extending the computational domain by one cell and using an upstream scheme for w+ in the last cell.
b) the idea is that the solution must be smooth if there is no reflection.

Equation and dispersion relation
We consider now scalar linear constant coefficient equations

$$P(D_t, D_x)q = 0$$

where $P(u,v)$ is a polynomial in u and v. It is satisfied by the exponential function of x and t, $q = e^{\lambda t + ikx}$, if λ and k satisfy the *dispersion relation*

$$P(\lambda, ik) = 0$$

For hyperbolic equations we often write $i\omega$ for λ. For systems, the dispersion relation is the characteristic equation of the system matrix.

Ex.
1. The transmission line equation above : $\mathbf{q} = e^{i\omega t + ikx}\mathbf{r} = e^{ik(c_0 t + x)}\mathbf{r}$ gives

$$-\omega^2 + k^2/(lc) = 0 \text{ or } \omega = \pm k/\sqrt{lc}$$

from which the *phase speed* c_0 is read off: $c_0 = \omega/k = \pm 1/\sqrt{lc}$. Such waves are called *non*-dispersive because the speed does *not* depend on the wave-number.

2. The heat equation,

$$q_t = q_{xx}$$

has dispersion relation $\lambda = -k^2$, so the waves do not travel: they *dissipate*, i.e. the amplitude decays on the spot, faster for high wave numbers – the effect exploited in the multi-grid algorithm for elliptic problems.

3. The Schrodinger equation of quantum mechanics, for constant potential V

$$i\psi_t = \psi_{xx} - V\psi$$

has dispersion relation $-\omega = -k^2 - V$, phase speed $c_0 = \omega/k = k + V/k$ - dispersive indeed.

Equation and vonNeumann amplification factor *G*

$$\text{Differential equation}: P(D_t, D_x)q = 0$$
$$\text{von Neumann } G \text{ for the solution}: q = e^{\lambda t}e^{ikx}$$
$$q(t+\Delta t)/q(t) = G(k,\Delta t), \text{ where } P(\lambda, ik) = 0 \text{ (the dispersion relation)}$$
$$\ln G/\Delta t = \lambda, \lambda \equiv \partial/\partial t, ik \equiv \partial/\partial x$$

$$G \text{ for the time-discrete scheme}: \quad \text{Shift operator } E_t : E_t f(t) = f(t+\Delta t), etc.$$
$$\text{Scheme}: \rho(E_t, E_x)Q_m^n = 0 : \rho(G, e^{ik\Delta x}) = \rho(e^{\lambda\Delta t}, e^{ik\Delta x}) = 0$$

So: expand the expression as a power series in Δt, interpret as terms in a differential equation by the equivalence $\lambda \equiv \partial/\partial t, ik \equiv \partial/\partial x$

Example
The one-sided scheme, advection equation $q_t + aq_x = 0$, $a > 0$

$$Q_m^{n+1} - Q_m^n + \sigma\left(Q_m^n - Q_{m-1}^n\right) = 0, \sigma = a\Delta t/\Delta x$$
$$0 = G - 1 + \sigma\left(1 - E_x^{-1}\right) = e^{\lambda\Delta t} - 1 + \sigma\left(1 - e^{-ik\Delta x}\right) = e^{\lambda\Delta t} - 1 + \sigma\left(1 - e^{-ika\Delta t/\sigma}\right)$$

First, not so successful attempt:

Expand the "dispersion relation" as power series in Δt:

$$\Delta t^0 : 0$$

$$\Delta t^1 : \lambda + ika : D_t + aD_x$$

$$\Delta t^2 : \lambda^2 / 2 + k^2 a^2 /(2\sigma) : \frac{1}{2}\left(D_t^2 - a^2 D_x^2 / \sigma\right)$$

So, the upstream scheme is a first order approximation to the advection equation, as it should, and a second order approximation to

$$u_t + au_x + \frac{\Delta t}{2}\left(u_{tt} - a^2 / \sigma u_{xx}\right) = 0$$

This does not clarify things much, because the highest order time derivative is now multiplied by the infinitesimally small Δt.

Second, successful attempt:
Expand λ as power series in *ik*. This means looking for a differential equation of the form

$$D_t q = \sum_{0 \le j} a_j D_x^j q$$

We have

$$0 = e^{\lambda \Delta t} - 1 + \sigma\left(1 - e^{-ika\Delta t / \sigma}\right) \Rightarrow \lambda = \ln\left(1 - \sigma\left(1 - e^{-ika\Delta t / \sigma}\right)\right) / \Delta t$$

Taylor expansions of Taylor expansions tax one's patience and accuracy. Symbolic algebra systems like Maple (e.g. inside Matlab) will perform power series expansions (in terms of Δt) in a second, and using `Dx` for *d/dx* = *ik* Maple expands λ into

```
Dtexp =
(-Dx*a*dt+(1/2/s*Dx^2*a^2-1/2*Dx^2*a^2)*dt^2+(-
1/6/s^2*Dx^3*a^3+1/6*Dx^3*a^3/s-1/3*Dx^3*a^3*(-1+s)/s)*dt^3)/dt
```

which needs cleaning up … the `simplify` function does not help, so manually:
Carry out the division by `dt` and factor out common factors:

```
Dtexp =
-Dx*a+ ...
1/2*dt*a^2*(1/s-1)*Dx^2 + ...
1/6*dt^2*a^3(-1/s^2+1/s-2*(-1+s)/s)*Dx^3
```

so the modified equation becomes, to second order

$$u_t = -au_x + \frac{\Delta t}{2} a^2 (1/\sigma - 1) u_{xx} + \frac{\Delta t^2}{2} a^3 \left(3/\sigma - 2 - 1/\sigma^2\right) u_{xxx} + O(\Delta t^3)$$

which says that:
1. the upstream scheme is a first order, dissipative approximation to the advection equation,
2. $\sigma < 1$ is necessary for the second derivative term to be dissipative
3. the third derivative term does not provide much information.

so it turns out we need compute only the first term of the truncation error. But of course one has to know which power of Δt that is …