Homework problem 2

Deadline February 20.

Linear and nonlinear hyperbolic systems

In this exercise we shall investigate the relation between a non-linear problem and the corresponding linearized system. In particular we will see how well linear analysis predicts the behavior of the nonlinear problem.
Shallow water flow over a horizontal bottom is modeled by

$$h_t + (hv)_x = 0 \qquad \text{(conservation of volume)}$$
$$v_t + vv_x + gh_x = 0 \; \text{(force balance in } x) \qquad (1)$$
$$\text{on } (x,t) \in [0,L]\times[0,\infty)$$

with boundary conditions $v(0) = v(L) = 0$, and initial conditions

$$h(x,0) = H + \varepsilon e^{-(x-L/2)^2/w^2}$$
$$v(x,0) = 0 \qquad (2)$$

We shall take L = 10m, H = 1m, and g = 9.61 (m/s). The width w of the water hill is 0.4 m and ε will be varied.

0. Conservation form

The mass balance equation is in conservation form, but not the momentum. Introduce $m = hv$ as a new variable instead of v and derive the flux function F in

$$m_t + F(m,h)_x = 0 \qquad (3)$$

h and m are the proper quantities that will be conserved, see the discussion in L. Of course, the smooth solutions, wave speeds, etc., are the same no matter how the equations are written. The quasi-linear form (1) is often most convenient for linearization.

1. Numerical Solution

To begin with let ε = 0.1 and solve numerically using the Lax-Friedrichs method. Use ghost cells at the boundaries. Prescribe values there by the technique of decomposing into characteristic variables *or* by the procedure described in L. Ch 7 for solid walls. Choose Δt and Δx after making numerical experiments with different discretization parameters. Use Δt/Δx as large as possible, without violating stability. Make plots showing wave propagation and reflections at the boundaries. Compute at least until waves have been reflected at both boundaries and crossed each other. Run the program again with larger values of ε (= 0.4, 0.8, 1.2, ….). How does the solution change? (Wave shape, amplitude, speed; "wiggles"). You may have to adjust the time-step to ensure stability.

2. Linearization

Choose a relevant constant state and derive the linearized problem at that state. Don't forget initial and boundary conditions. Show that the linear problem is hyperbolic.

3. Analysis of Linear Problem

The linear problem can be solved analytically. Determine the solution of the linear problem at for instance t = 1. Discuss how information propagates, if the boundary conditions cause reflections and when reflected waves will appear. Compare with the numerical results for the non-linear case.

4. Non-Reflecting boundary condition

Derive boundary conditions for the linear problem that *do not* cause reflections. Formulate corresponding conditions for the non-linear case. Implement the conditions in your program using either the technique of characteristic variables or by simply extrapolating all variables at the boundary (as described in chapter 7). How well does the method work? Try to measure the size of the reflection. (See lecture notes L3 – L4).