L1

Galerkin orthogonality property: (W) − (G)

$$\int_0^1 (u-U)'\, v'\, dx = 0 \quad \forall v \in V_h \qquad \text{(G.O.)}$$

(Since all functions in V_h are also in V: $V_h \subset V$)

Norms: measuring the size of a function (compare absolute value for a number)

L_2-norm: $\|f\| = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}$

Energy-norm: $\|v\|_E = \left(\int_0^1 |v'(x)|^2 dx \right)^{1/2}$

How large is the error $u-U$ in $\|\cdot\|_E$?

$$\|(u-U)'\|^2 = \int_0^1 (u-U)'(u-U)'\, dx = \left[\pm \int_0^1 v'\, dx \right]$$

$$= \int_0^1 (u-U)'(u-v)'\, dx + \underbrace{\int_0^1 (u-U)'\underbrace{(v-U)'}_{\in V_h}\, dx}_{=\,0 \ \text{(G.O.)}}$$

$$= \int_0^1 (u-U)'(u-v)'\, dx \le \|(u-U)'\|\, \|(u-v)'\|$$

[Cauchy-Schwarz ineq. $\int_0^1 fg\, dx \le \|f\|\,\|g\|$]

$$\Rightarrow \quad \|(u-U)'\| \le \|(u-v)'\| \quad \forall v \in V_h$$

$$\|u-U\|_E \le \|u-v\|_E \quad \forall v \in V_h$$

Galerkin approx. optimal in energy norm!