Computing the residual using the discrete Laplacian Δ_h

For a given $w \in V$, let $\Delta_h w$ be the unique function in V_h s.t.

$$(*) \qquad -(\Delta_h w, v) = (\nabla w, \nabla v) \quad \forall v \in V_h$$

How to compute $\Delta_h U$?

$U = \sum_{j=1}^{M} \eta_j \phi_j$ with $\eta_j = U(N_j)$ nodal values

$$\Delta_h U \in V_h \Rightarrow \Delta_h U = \sum_{j=1}^{M} \zeta_j \phi_j$$

$$(*) \Rightarrow -\sum_{j=1}^{M} \zeta_j (\phi_j, \phi_i) = \sum_{j=1}^{M} \eta_j (\nabla\phi_j, \nabla\phi_i)$$

$$i = 1, \dots, M$$

Corresponds to linear system of equations:

$$-M\zeta = A\eta$$

with $\zeta = (\zeta_j)$ $\eta = (\eta_j)$ $M = (\phi_j, \phi_i)$ $A = (\nabla\phi_j, \nabla\phi_i)$

M mass matrix, A stiffness matrix

$$\Rightarrow \zeta = -M^{-1}A\eta \Rightarrow \Delta_h U = \sum_{j=1}^{M} \zeta_j \phi_j$$

$$\Rightarrow R(U) = f + \Delta U \approx f + \Delta_h U$$