Galerkin orthogonality : $(\nabla u - \nabla U, \nabla v) = 0 \quad \forall v \in V_h$

A priori error estimate of error $e = u - U$:

$$\|\nabla e\|^2 = (\nabla e, \nabla(u-U)) = (\nabla e, \nabla(u-U)) + (\nabla e, \nabla(\underbrace{U-v}_{\in V_h}))$$
$$= (\nabla e, \nabla(u-v)) \leq \|\nabla e\| \, \|\nabla(u-v)\|$$

$\Rightarrow \|\nabla e\| \leq \|\nabla(u-v)\| \quad \forall v \in V_h$ ($U \in V_h$ optimal w.r.t. energy norm)

Interpolation error estimate : choose $v = \pi_h u$

$$\Rightarrow \|\nabla(u-U)\| \leq \|\nabla(u - \pi_h u)\| \leq C_i \|h D^2 u\|$$

A posteriori error estimate :

$$\|\nabla e\|^2 = (\nabla(u-U), \nabla e) = (\nabla u, \nabla e) - (\nabla U, \nabla e)$$
$$= (f, e) - (\nabla U, \nabla e) = (f, e - \tilde{\pi}_h e) - (\nabla U, \nabla(e - \tilde{\pi}_h e))$$

with $\tilde{\pi}_h e \in V_h$ interpolant based on weighted averages around node.

Divide integrals over elements K & Green's formula

$$\|\nabla e\|^2 = \sum_K \int_K (f + \Delta U)(e - \tilde{\pi}_h e)\, dx - \sum_K \int_{\partial K} \frac{\partial U}{\partial n_K}(e - \tilde{\pi}_h e)\, ds$$

$\frac{\partial U}{\partial n_K}$ with respect to outward normal n_K of element boundary ∂K