Ex: Heat equation

$$\begin{cases} \dot{u} - \Delta u = f & (x,t) \in \Omega \times (0, T] \\ u = 0 & x \in \partial\Omega \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

$$(Av, v) = (-\Delta v, v) = (\nabla v, \nabla v) = \|\nabla v\|^2 \geq 0$$

$\Rightarrow$ Heat equation is parabolic!

Stability/Energy estimates: mult. by function & integrate

1. Mult. by u & integrate $\Rightarrow \int_\Omega (\dot{u} - \Delta u) u \, dx = (\dot{u} - \Delta u, u) = 0$
(assume that $f = 0$)

$$\Rightarrow (\dot{u}, u) - (\Delta u, u) = 0 \Leftrightarrow \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = 0$$

$$\left(\frac{d}{dt}\|u\|^2 = \frac{d}{dt}\int_\Omega |u|^2 dx = \frac{d}{dt}\int_\Omega u_1^2 + u_2^2 + u_3^2 \, dx = \int_\Omega 2u_1\dot{u}_1 + \dots + 2u_3\dot{u}_3 = 2(\dot{u}, u)\right)$$

Integrate in time $\Rightarrow$ $\boxed{\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2 dt = \|u_0\|^2}$ (1)

$2\int_0^t \|\nabla u\|^2 dt$ increase with time t

$\Rightarrow \|u(t)\|^2$ decrease with time t

(dissipation of energy $\|u(t)\|^2$)