Divergence form $\nabla\cdot(\beta u)$

Assume: $\varepsilon\,\partial_n u = 0$ on $\Gamma \times I$ (insulation)

$\beta \cdot n = 0$ on $\Gamma \times I$ (no convection through the boundary)

$f = 0$ (no heat source)

$\alpha = 0$ (no absorption)

$$\Rightarrow \frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \dot u\,dx = \int_\Omega \big(\nabla\cdot(\varepsilon\nabla u) - \nabla\cdot(\beta u)\big)\,dx$$

[Gauss Thm: (13.13) $\int_\Omega \nabla\cdot v\,dx = \int_\Gamma v\cdot n\,ds$]

$$= \int_\Gamma \big(\varepsilon\,\partial_n u - \underbrace{(\beta u)\cdot n}_{u(\beta\cdot n)}\big)\,ds = 0 \Rightarrow \int_\Omega u\,dx \text{ conserved!}$$

Non-divergence form: $\beta\cdot\nabla u$

$$\Rightarrow \frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \dot u\,dx = \int_\Omega \nabla\cdot(\varepsilon\nabla u) - \beta\cdot\nabla u\,dx$$

$$= \int_\Omega \underbrace{\nabla\cdot(\varepsilon\nabla u) - \nabla\cdot(\beta u)}_{=0 \text{ as above}} + (\nabla\cdot\beta)\,u\,dx$$

$$= \int_\Omega (\nabla\cdot\beta)\,u\,dx$$

Conservative only if $\nabla\cdot\beta = 0$

(β divergence free)