Streamline diffusion method

(a) introduction of weighted least squares terms.
(b) ——— " ——— artificial viscosity based on residual.

(D) $Au = f$ (differential equation)

(G) Find $U \in V_h$: $(AU, v) = (f, v) \quad \forall v \in V_h$

(LS) Find $U \in V_h$: $\|AU - f\|^2 = \min_{v \in V_h} \|Av - f\|^2$

corresponds to $(AU, Av) = (f, Av) \quad \forall v \in V_h$

(GLS): Find $U \in V_h$: $(AU, v) + (\delta AU, Av) = (f, v) + (\delta f, Av) \; \forall v \in V_h$

(weighted combination of (G) & (LS): $\delta > 0$)

Petrov-Galerkin method: $(V_h \neq \hat{V}_h)$

Find $U \in V_h$: $(AU, v + \delta Av) = (f, v + \delta Av) \quad \forall v \in V_h$

(Sd): Find $U \in V_h$: $(AU, v + \delta Av) + (\hat{\varepsilon} \nabla U, \nabla v) = (f, v + \delta Av)$

$\hat{\varepsilon} = \gamma_1 h^2 |R(U)|$ (artificial viscosity) $\quad \forall v \in V_h$

Assume $(Av, v) \geq c\|v\|^2 \quad c > 0$ ~~[illegible]~~

Set $v = U$ in (Sd) $\Rightarrow (AU, U) + (\delta AU, AU) + (\hat{\varepsilon} \nabla U, \nabla U) = (f, U) + (\delta f, AU)$

$$\Rightarrow c\|U\|^2 + \|\sqrt{\delta} AU\|^2 + \|\sqrt{\hat{\varepsilon}} \nabla U\|^2 \leq \|f\| \|U\| + \|\sqrt{\delta} f\| \|\sqrt{\delta} AU\|$$

$$\leq \frac{c}{2}\|U\|^2 + \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta} AU\|^2 + \frac{1}{2}\|\sqrt{\delta} f\|^2$$

$$\Rightarrow \frac{c}{2}\|U\|^2 + \frac{1}{2}\|\sqrt{\delta} AU\|^2 + \|\sqrt{\hat{\varepsilon}} \nabla U\|^2 \leq \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta} f\|^2$$

$$\Rightarrow \left(\|U\| + \|\sqrt{\delta} AU\| + \|\sqrt{\hat{\varepsilon}} \nabla U\|\right)^2 \leq C\|f\|^2$$

$$\Rightarrow \boxed{\|U\| + \|\sqrt{\delta} AU\| + \|\sqrt{\hat{\varepsilon}} \nabla U\| \leq C\|f\|}$$