Ex: Navier-Stokes equations (NS)

$$R(\hat{u}) = \begin{bmatrix} \dot{u} + u\cdot\nabla u - \nu\Delta u + \nabla p - f \\ \nabla\cdot u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\hat{u} = (u,p)$; $u : \Omega\times I \to \mathbb{R}^3$ velocity vector
$p : \Omega\times I \to \mathbb{R}$ scalar pressure

ν ; viscosity

- Existence & Uniqueness of pointwise solution
  $\hat{u}=(u,p)$: $R(\hat{u})=0$, unknown.
  ($1 million Clay Prize problem)
- Existence (not unq.) of weak solution
  known (Leray 1934): Find $\hat{u}\in\hat{V}$:
  $$(R(\hat{u}),\hat{v})_Q = 0 \quad \forall \hat{v}=(v,q)\in\hat{V}$$

$\hat{V} \subset H^1(Q)$; $Q=\Omega\times I$ space-time domain.

$$(R(\hat{u}),\hat{v}) = (\dot{u},v) + (u\cdot\nabla u, v) - (\nabla\cdot v, p) + (\nabla\cdot u, q) + (\nu\nabla u, \nabla v) \quad \text{(assume } f=0\text{)}$$

Galerkin FEM: Find $\hat{U}\in\hat{V}_h$: $(R(\hat{U}),\hat{v})=0$ $\forall \hat{v}\in\hat{V}_h$

Energy est.: $\frac{1}{2}\|U(t)\|^2 + \int_0^t \|\sqrt{\nu}\,\nabla U\|^2 dt = \frac{1}{2}\|U(0)\|^2$
(same for exact sol.)

GLS: Find $\hat{U}\in\hat{V}_h$: $(R(\hat{U}),\hat{v}) + (\delta R(\hat{U}), R(\hat{v})) = 0$ $\forall \hat{v}\in\hat{V}_h$

$$\frac{1}{2}\|U(t)\|^2 + \int_0^t \|\sqrt{\nu}\nabla U\|^2 dt + \int_0^t \|\sqrt{\delta}\,R(\hat{U})\|^2 dt = \frac{1}{2}\|U(0)\|^2$$