Compute output $\pi(\hat{u}) = (\hat{u}, \hat{\psi})$

Ex. $\pi(\hat{u})$ drag of a car, lift of airplane.

Introduce dual problem:

$$\begin{cases} -\dot{\varphi} - u \cdot \nabla \varphi + \nabla U \cdot \varphi + \nabla \theta = \psi_1 \\ \nabla \cdot \varphi = \psi_2 \end{cases}$$

$$\hat{\varphi} = (\varphi, \theta), \quad \hat{\psi} = (\psi_1, \psi_2)$$

A posteriori error estimate:

$$|\pi(\hat{u}) - \pi(\hat{U})| = |(\hat{u} - \hat{U}, \hat{\psi})| = \dots = |(R(\hat{U}), \hat{\varphi})|$$

$$= |(R(\hat{U}), \hat{\varphi} - \pi_h \hat{\varphi}) - (SR(\hat{U}), R(\pi_h \hat{\varphi}))|$$

$$\leq C_i S(\hat{\varphi}) \|h R(\hat{U})\| + |SD_S(\hat{U}; \pi_h \hat{\varphi})|$$

$S(\hat{\varphi})$ stability factor: $S(\hat{\varphi}) = \|D\hat{\varphi}\|_Q$

- Rewrite as sum over all elements:

$$|\pi(\hat{u}) - \pi(\hat{U})| \leq \sum_K \varepsilon_K = \sum_K C_i \|h R(\hat{U})\|_K \|D\hat{\varphi}_h\| + |SD_f(\hat{U}; \hat{\varphi}_h)|$$

- Adaptive algorithm to compute $\pi(\hat{u})$