# 2D1260 Finite Element Methods: Written Examination

## Wednesday 2002-12-18, kl 8-13

**Aids:** None. **Time:** 5 hours.

Answers may be given in English or Swedish.

Please note that answers should be explained and calculations shown unless the question states otherwise. A correct answer without explanation can thus be left without points.

---

**(5)** **1.** Consider the boundary value problem

$$-(x\,u')' + 3u = x^2, \qquad 1 < x < 2$$

$$u'(1) = 1,\ u(2) = 0$$

Approximate the solution by a quadratic polynomial using Galerkins method.

**(5)** **2.** Use three linear finite elements to solve the previous problem approximately. The discretization points are $x = [1,\ 1.3,\ 1.5,\ 2]$. You may use a 1-point quadrature for the integrals. It is not necessary to solve the resulting final system of equations.

**(5)** **3.** Let the differential equation

$$-\Delta u = 1 \quad \text{on } \Omega$$

be given on the quadrilateral domain with vertices $(1, 2)$, $(3, 1)$, $(4, 2)$ and $(2, 4)$. The boundary values are

$$u = x \text{ on the boundary between } (3, 1) \text{ and } (4, 2)$$

$$\frac{\partial u}{\partial n} = 0 \text{ on the other boundaries}$$

Solve the problem using FEM and two linear finite elements obtained by subdividing $\Omega$ along the diagonal connecting $(1, 2)$ and $(4, 2)$.

**N.B.** *The exam continues on the next page.*

**(5)** **4.** Derive a weak formulation of the problem

$$-\nabla(k\ \nabla u) + \gamma\ u = f \qquad \text{on } \Omega$$

where $\Omega$ is the first quarter of the unit circle:

$$\Omega = \begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi, \end{cases} \qquad \text{with} \qquad \begin{cases} 0 < r < 1 \\ 0 < \varphi < \pi/2 \end{cases}$$

and the boundary conditions are

$$\begin{aligned} u = x, &\qquad \text{when } y = 0 \\ k\,\frac{\partial u}{\partial x} = 1, &\qquad \text{when } x = 0 \\ u = 2 - x, &\qquad \text{on the curved boundary} \end{aligned}$$

**(3)** **5.** Describe 4 typical finite elements.

**(2)** **6.** Describe an *a-priori* error estimate. Describe an *a-posteriori* error estimate. Comment on similarities and differences in the estimations.

*Good luck*

*and*

*Merry Christmas & Happy New Year!*

*NINNI*

## 2D1260 FEM 2002-12-18: Short solutions

**1.** Obtain a weak form: Find $u$ such that

$$\int_1^2 (-(xu')' + 3u)\ v\ dx = \int_1^2 x^2\ v\ dx$$

for any $v$ such that $v(2) = 0$ (since Dirichlet BC at $x = 2$). Do partial integration to lower order of derivatives:

$$\begin{aligned}\int_1^2 -(xu')'v\ dx &= \big[-(xu')\,v\big]_1^2 - \int_1^2 (-xu')\ v'\ dx \\ &= -2u'(2)v(2) + 1u'(1)v(1) + \int_1^2 xu'\ v'\ dx \\ &= v(1) + \int_1^2 xu'\ v'\ dx\end{aligned}$$

since $v(2) = 0$ and $u'(1) = 1$. Leading to the weak form: Find $u$ such that for any $v$ with $v(2) = 0$

$$\int_1^2 xu'v' + 3uv\ dx = -v(1) + \int_1^2 x^2\ v\ dx$$

The ansatz should be a second order polynomial (3 coefficients) with one extra requirement (loose one, leaving two unknown coefficients). General second order polynomial is

$$p_2(x) = c_1 + c_2 x + c_3 x^2$$

The demand $p(2) = 0$ leads to the condition $c_1 + 2c_2 + 4c_3 = 0$. Choose two parameters, leading to

$$p(x) = \alpha_1 v_1(x) + \alpha_2 v_2(x)$$

If we 'choose' $\alpha_1 = c_2$ and $\alpha_2 = c_3$ we thus obtain

$$c_1 = -2c_2 - 4c_3 = -2\alpha_1 - 4\alpha_2$$

giving

$$\begin{aligned}p(x) &= (-2\alpha_1 - 4\alpha_2) + \alpha_1 x + \alpha_2 x^2 \\ &= \alpha_1(x-2) + \alpha_2(x^2 - 4) \\ &= \alpha_1 v_1(x) + \alpha_2 v_2(x)\end{aligned}$$

The Galerkin method thus means testing the weak formulation with $U = \alpha_1 v_1 + \alpha_2 v_2$ and $v = v_1$ and $v = v_2$. This leads to the $2 \times 2$ system of equations

$$\left( \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \right) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

where

$$\begin{aligned} S_{ij} &= \int_1^2 x\, v_i'\, v_j'\ dx \\ Q_{ij} &= \int_1^2 3\, v_i\, v_j\ dx \\ B_i &= -v_i(1) \\ F_i &= \int_1^2 x^2\, v_i\ dx \end{aligned} \qquad \text{with} \qquad \begin{aligned} v_1 &= x - 2 \\ v_2 &= x^2 - 4 \end{aligned}$$

Calculus gives

$$\left(\begin{bmatrix} 3/2 & 14/3 \\ 14/3 & 15 \end{bmatrix} + \begin{bmatrix} 1 & 13/4 \\ 13/4 & 53/5 \end{bmatrix}\right)\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -11/12 \\ -47/15 \end{bmatrix}$$

giving $\alpha_1 \approx 2.4042$ and $\alpha_2 \approx -0.7487$.

**2.** The weak formulations is of course the same as above. Use linear base functions. Use standard element

$$\varphi_1 = 1 - \xi, \qquad \varphi_2 = \xi$$

The Jacobian becomes $J = L_k$ The 3 elements have lengths $L_1 = 0.3$, $L_2 = 0.2$ and $L_3 = 0.5$. The $S$-matrix becomes

$$S^{(k)} = \int_0^1 \hat{x}\left(\frac{1}{L_k}\right)^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} L_k \, d\xi = \frac{m_k}{L_k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

using mid-point quadrature with $m_k$ as the midpoint of the interval (in true coordinates!). The **Q**-matrix becomes

$$\mathbf{Q}_{ij}^{(k)} = \int_0^1 3\varphi_i\, \varphi_j \det J \, d\xi = 3L_k \int_0^1 \varphi_i\, \varphi_j \, d\xi \quad \Rightarrow \quad \mathbf{Q}^{(k)} = 3L_k\mathbf{M}^{(k)} = 3L_k\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = L_k\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

where **M** is the mass matrix.

$v(1)$ only affect equation 1 ($v = \varphi_1$), where $\varphi(1) = 1$. Thus $B_1 = -1$ leading to the vector $B = (-1, 0, 0, 0)^T$. Finally the element load vector is calculated

$$f_i^{(k)} = \int_0^1 \hat{x}^2 \varphi_i(\xi) \det J \, d\xi = (\text{midpoint}) = m_k^2 \cdot 0.5 \cdot L_k, \quad i = 1, 2$$

since both basis funcstions have the value 0.5 at the midpoint of the interval. Assembling the **S**-matrix ($m_1/L_1 = 0.5,\ m_2/L_2 = 2,\ m_3/L_3 = 1.5$) and **Q**-matrix: (Error in calc! $m_1/L_1$ should been 1.15/0.3 not 0.15/0.3, and so on. Numbers will be corrected soon...)

$$\mathbf{S} = \begin{bmatrix} 0.5 & -0.5 & 0 & 0 \\ -0.5 & 0.5+2 & -2 & 0 \\ 0 & -2 & 2+1.5 & -1.5 \\ 0 & 0 & -1.5 & 1.5 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 0.3 & 0.15 & 0 & 0 \\ 0.15 & 0.3+0.2 & 0.1 & 0 \\ 0 & 0.1 & 0.2+0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

and the **f**-vector ($m_k^2 \cdot L_k/2$ equals 0.003375, 0.016 and 0.140625 respectively:

$$\mathbf{f} = (\,0.003375 \quad 0.003375 + 0.016 \quad 0.016 + 0.140625 \quad 0.140625\,)^T$$

leading to the generalised stiffness matrix and load vector

$$\tilde{\mathbf{S}} = \mathbf{S} + \mathbf{Q} = \begin{bmatrix} 0.8 & -0.35 & 0 & 0 \\ -0.35 & 3 & -1.9 & 0 \\ 0 & -1.9 & 4.2 & -1.25 \\ 0 & 0 & -1.25 & 2 \end{bmatrix} \qquad \tilde{\mathbf{f}} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.003375 \\ 0.019375 \\ 0.156625 \\ 0.140625 \end{bmatrix} = \begin{bmatrix} -0.996625 \\ 0.019375 \\ 0.156625 \\ 0.140625 \end{bmatrix}$$

Finally we adjust the stiffness matrix and load vector for known values (the Dirichlet boundary conditions), here the last value:

$$\tilde{\mathbf{S}} = \begin{bmatrix} 0.8 & -0.35 & 0 & 0 \\ -0.35 & 3 & -1.9 & 0 \\ 0 & -1.9 & 4.2 & -1.25 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \tilde{\mathbf{f}} = \begin{bmatrix} -0.996625 \\ 0.019375 \\ 0.156625 \\ 0 \end{bmatrix}$$

**3.** The DE $-\Delta u = 1$ with zero Neumann BC:s has weak formulation

$$\int_\Omega -\Delta u v dx = \int_\Omega \nabla u \nabla v dx = \int_\Omega 1 v dx$$

Using the standard triangle element

$$\phi_1 = 1 - \xi - \eta, \qquad \phi_2 = \xi, \qquad \phi_3 = \eta$$

we have

$$B = \begin{bmatrix} \frac{\partial \phi_1}{\partial \xi} & \frac{\partial \phi_2}{\partial \xi} & \frac{\partial \phi_3}{\partial \xi} \\ \frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \frac{\partial \phi_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The element stiffness matrix will be

$$S_k = \int_0^1 \int_0^\eta (J^{-1}B)^T (J^{-1}B) \det J \; d\xi d\eta = (J^{-1}B)^T (J^{-1}B) \det J \; \frac{1}{2}$$

if the Jacobian is a constant over each element (which it is with linear basis functions). The components of the element load vector will be

$$f_i = \int_0^1 1 \cdot \phi_i(\xi, \eta) \det J \; d\xi d\eta = \frac{1}{6} \det J$$

The Jacobian is obtained by the iso-parametric mapping $\bar{x}(\xi, \eta) = \sum_{i=1}^3 \phi_i(\xi, \eta) \cdot \bar{x}_i$, here:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= (1 - \xi - \eta) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \xi \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \eta \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \xi \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} + \eta \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \end{aligned}$$

The Jacobian is thus

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$

I now number the nodes, anticlockwise, starting with $(1, 2)$ as number 1 and call the lower triangle element 1.
On element 1 the nodes are 1, 2 and 3. The Jacobian is thus

$$J = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \qquad \text{with} \qquad \det J = 3$$

giving the local stiffness matris and element vector

$$S^{(1)} = \begin{bmatrix} 1/3 & -1/2 & 1/6 \\ -1/2 & 3/2 & -1 \\ 1/6 & -1 & 5/6 \end{bmatrix} \qquad \text{and} \qquad F^{(1)} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

On element 2 the nodes are 1, 3 and 4. The Jacobian is thus

$$J = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \qquad \text{with} \qquad \det J = 6$$

giving the local stiffness matris and element vector

$$S^{(2)} = \begin{bmatrix} 2/3 & -1/6 & -1/2 \\ -1/6 & 5/12 & -1/4 \\ -1/2 & -1/4 & 3/4 \end{bmatrix} \qquad \text{and} \qquad F^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The global matrices is thus

$$S = \begin{pmatrix} 1/3+2/3 & -1/2 & 1/6-1/6 & -1/2 \\ -1/2 & 3/2 & -1 & 0 \\ 1/6-1/6 & -1 & 5/6+5/12 & -1/4 \\ -1/2 & 0 & -1/4 & 3/4 \end{pmatrix} \qquad \text{with} \qquad F = \begin{pmatrix} 1/2+1 \\ 1/2 \\ 1/2+1 \\ 1 \end{pmatrix}$$

Adjust for the known Dirichlet boundary conditions in nodes 2 and 3 ($u_2 = 3$ and $u_3 = 4$)

$$\begin{pmatrix} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/2 & 0 & -1/4 & 3/4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3 \\ 4 \\ 1 \end{pmatrix}$$

The solution is

$$u = \begin{pmatrix} 6.5 & 3 & 4 & 7 \end{pmatrix}^T$$

**4.** For the weak formulation we will use Gauss' Theorem which is

$$\int_\Omega \nabla \cdot \bar{w} \; d\Omega = \int_\Gamma \bar{w} \cdot \hat{n} \; dS$$

where $\Gamma$ is the close surface surrounding $\Omega$. From the rule of derivation of products we obtain

$$\nabla \cdot (\alpha \bar{w}) = \alpha(\nabla \cdot \bar{w}) + \bar{w} \cdot (\nabla \alpha)$$

We thus have

$$\int_\Omega \nabla \cdot (\bar{q}\; v) \; d\Omega = \int_\Omega (\nabla \cdot \bar{q})\, v \; d\Omega + \int_\Omega \bar{q} \cdot \nabla v \; d\Omega \;\Rightarrow\; \int_\Omega (\nabla \cdot \bar{q})\, v \; d\Omega = \int_\Gamma \hat{n} \cdot \bar{q}\; v \; ds - \int_\Omega \bar{q} \cdot \nabla v \; d\Omega$$

The differential equation read

$$-\nabla(k\; \nabla u) + \gamma\; u = f \qquad \text{on } \Omega$$

a weak formulation then is

$$\int_\Omega (-\nabla(k\; \nabla u))\; v \; d\Omega + \int_\Omega \gamma\; u\; v \; d\Omega = \int_\Omega f\; v \; d\Omega$$

The first integral is changed using Gauss' Theorem with $\bar{q} = -k\nabla u$:

$$\int_\Omega (-\nabla(k\; \nabla u))\; v \; d\Omega = \int_\Gamma \hat{n} \cdot (-k\; \nabla u)\; v \; ds - \int_\Omega (-k\; \nabla u) \cdot \nabla v \; d\Omega$$

The boundary $\Gamma$ is split in three parts, $\Gamma_1$ where $y = 0$, $\Gamma_2$ where $x = 0$ and the curved boundary $\Gamma_3$. On $\Gamma_1$ and $\Gamma_3$ we have Dirichlet boundary conditions, thus we have $v = 0$ there. On $\Gamma_2$ we have a non-zero Neumann boundary conditions, thus this is the only part of $\int_\Gamma$ which remains.

$$\int_\Gamma \hat{n} \cdot (-k\; \nabla u)\; v \; ds = \int_{\Gamma_1} (+1)\; v \; ds$$

This leads to the following weak formulation: Find $u$ such that

$$\int_\Omega k\; \nabla u \cdot \nabla v \; d\Omega + \int_\Omega \gamma\; u\; v \; d\Omega = \int_\Omega f\; v \; d\Omega - \int_{\Gamma_1} v \; ds$$

for all $v$ such that $v = 0$ on $\Gamma_1$ and $\Gamma_3$ (and $u$ and $v$ must be continuous and once differentiable).

**5.** Typical finite elements have basis functions that are 1 in 'their own' node and 0 at all the others. Typical basis functions are the Lagrange polynomials.
Examples of typical finite elements are

- 1D, with one node at each end of the interval and linear basis functions.
- 1D, with one node at each end of the interval and one in the interior , with quadratic basis functions.
- 2D triangular element, with one node at each corner and linear basis functions.
- 2D quadrilateral element, with one node at each corner and bilinear basis functions.
- 2D 8-node serendipity element, se textbook for details.
- 2D $M \times N$-node Lagrange element, se textbook for details.
- 3D $M \times N \times Q$-node Lagrange element, se textbook for details.

**6.** An **a-priori error estimate** is done before the actual calculation is done. It is formulated on the exact solution and known parameters such as the grid size. Typical form of the estimate is (for 2nd order PDE):

$$||u - U||_E \leq C_k h^k ||D^{k+1}u||_{L^2(\Omega)}$$
$$||u - U||_{L^2(\Omega)} \leq \tilde{C}_k h^{k+1} ||D^{k+1}u||_{L^2(\Omega)}$$

where $u$ is the exact solution, $k+1$ times differentiable, $U$ is the FEM solution, $C_k$ and $\tilde{C}_k$ are constants, independent of $h, k$ and $u$ (but dependent on $\Omega$ and mesh properties), $h$ is a mesh size parameter, $D^{k+1}u$ are all $(k+1)$ derivatives of $u$, and $k$ is the order of the piecewise polynomials used.
The estimate gives a qualitative picture on the speed of convergence if $h \to 0$. (i.e. with grid refinement).

An **a-posteriori error estimate** is done after the actual calculation is done. It is based on the computed FEM solution $U$ (and qualitative information of the exact solution $u$). Typical form of the estimate uses the residual estimation

$$||e|| \leq C^i C^s ||h^k R(U)||$$

where $C^i$ is an interpolation constant (must be estimated, e.g. by refining the grid once or using a known solution) $C^s$ is a stability constant (must generally be estimated, or as above), and $R(U)$ is the residual. A coarse but often used approximation is

$$||e|| \approx ||U^{\text{coarse grid}} - U^{\text{fine grid}}||$$

which however generally underestimates the error. Another common a-posteriori error estimate is to use a sequence of finer grids and see how $U$ changes with grid size and do a Richardson extrapolation.

(Actually Richardson extrapolation in 2D and higher dimensions is risky - if there is a singularity at the boundary of the domain there is no asymptotic expression. Also, comparision bewtween coarser and finer grids are often too cumbersome in 3D and higher.)