2D1260 Finite Element Methods: Written Examination

Saturday 2003-02-15, kl 8-13

Aids: None. **Time:** 5 hours.

Answers may be given in English or Swedish.

Please note that answers should be explained and calculations shown unless the question states otherwise. A correct answer without explanation can thus be left without points.

Using a desk calculator is not allowed. It is thus allowed to leave simple expressions unsimplified which could easily be calculated on a simple desk calculator. For example $\alpha = 0.3 \cdot 0.15^3 \cdot 0.7$

---

(5) **1.** Consider the boundary value problem

$$-(x^2\, u')' - u = x^2, \qquad 0.1 < x < 0.8$$

$$u(0.1) = 1,\ u'(0.8) = 4$$

Approximate the solution by a quadratic polynomial using Galerkins method.

(You may use a 1-point quadrature for the integrals. It is not necessary to solve the resulting final system of equations.)

(5) **2.** Solve the differential problem above using two second order finite elements. The element endpoints are $x = [0.1,\ 0.5,\ 0.8]$. You may use a 1-point quadrature for the integrals. It is not necessary to solve the resulting final system of equations.

(Using linear finite elements will not give full points in this exercise.)

(5) **3.** Let the differential equation

$$-\nabla \cdot (x\, \nabla u) = 2 \quad \text{on } \Omega$$

be given on the quadrilateral domain with vertices $(2, 1)$, $(3, 1)$, $(3, 4)$ and $(2, 4)$. The boundary values are

$$u = x^2 \qquad \text{where } y = 1$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on the other boundaries}$$

Solve the problem using FEM and two linear finite elements obtained by subdividing Ω along the diagonal connecting $(2, 1)$ and $(3, 4)$.

(You may use a 1-point quadrature for the integrals. It is not necessary to solve the resulting final system of equations.)

N.B. *The exam continues on the next page.*

(5) **4.** Derive a weak formulation of the 2D-problem

$$\beta \cdot \nabla u - \nabla \cdot (\varepsilon \, \nabla u) + \gamma \, u = f \qquad \text{on } \Omega$$

where Ω is the first quarter of the unit circle:

$$\Omega = \begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi, \end{cases} \qquad \text{with} \qquad \begin{cases} 0 < r < 1 \\ 0 < \varphi < \pi/2 \end{cases}$$

and the boundary conditions are

$$\begin{aligned} (\nabla u) \cdot n &= g, && \text{when } y = 0 \\ \frac{\partial u}{\partial y} &= h, && \text{when } x = 0 \\ u &= j, && \text{on the curved boundary} \end{aligned}$$

where f, g, h and j are smooth functions, $\beta = (\beta_x, \beta_y)$ is a field vector and $\varepsilon > 0$ a positive constant.

(5) **5.**

Figures missing: Grid 1= Mesh of quads Grid 2 = Mesh of triangles

a) Perform one standard refinement of grid 1 above.

b) Perform one standard refinement of grid 2 above.

c) What is the Delauney criteria for triangulation of a grid?

d) Does grid 2 above fulfill the Delauney criteria for triangulation? Show how you deduce your answer.

Good luck!

NINNI

2D1260 FEM 2003-02-15: Hints to solutions

No official solutions are made to "reexaminations", thus this is only a "working paper". Please beware of misprints. /Ninni

1. Obtain a weak form: Find u such that

$$\int_{0.1}^{0.8} (-(x^2u')' - u)\ v\ dx = \int_{0.1}^{0.8} x^2\ v\ dx$$

for any v such that $v(0.1) = 0$ (since Dirichlet BC at $x = 0.1$). Do partial integration to lower order of derivatives:

$$\begin{aligned}
\int_{0.1}^{0.8} -(x^2u')'v\ dx &= \big[-(x^2u')\,v\big]_{0.1}^{0.8} - \int_{0.1}^{0.8} (-x^2u')\ v'\ dx \\
&= -0.8^2u'(0.8)v(0.8) + 0.1^2u'(0.1)v(0.1) + \int_{0.1}^{0.8} xu'\ v'\ dx \\
&= -2.56\,v(0.8) + \int_{0.1}^{0.8} xu'\ v'\ dx
\end{aligned}$$

since $v(0.8) = 0$ and $u'(0.8) = 4$. Leading to the weak form: Find u such that for any v with $v(0.8) = 0$

$$\int_{0.1}^{0.8} xu'v' - uv\ dx = 2.56\,v(0.1) + \int_{0.1}^{0.8} x^2\ v\ dx$$

The ansatz should be a second order polynomial (3 coefficients). With one requirement on v we are left with two unknown coefficients. A general second order polynomial is

$$p_2(x) = c_1 + c_2x + c_3x^2$$

The testfunctions v should be zero at $x = 0.1$, thus a possible choice is to put the constant function as zero, the linear function as $v_1 = x - 0.1$ and the quadratic function as $v_2 = x^2 - 0.1^2$, giving $U = 1 + \alpha_1 \cdot v_1(x) + \alpha_2 \cdot v_2(x)$.

The Galerkin method means testing the weak formulation with $u = U$ and $v = v_1$ and $v = v_2$. This leads to the 2×2 system of equations

$$\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} - \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \right) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

where

$$\begin{aligned}
S_{ij} &= \int_{0.1}^{0.8} x^2\, v_i'\, v_j'\ dx \\
Q_{ij} &= \int_{0.1}^{0.8} v_i\, v_j\ dx \qquad \text{with} \qquad \begin{aligned} v_1 &= x - 0.1 \\ v_2 &= x^2 - 0.1^2 \end{aligned} \\
B_i &= 2.56 v_i(0.8) \\
F_i &= \int_{0.1}^{0.8} x^2\, v_i\ dx
\end{aligned}$$

Using mid-point quadrature, $x = (0.1 + 0.8)/2 = 0.45 = \hat{x}$, we have

$$S_{11} \approx 0.45^2 \cdot 1^2 \cdot 0.7\ (= 0.14175)$$

$$S_{12} \approx 0.45^2 \cdot 1 \cdot (2 \cdot 0.45) \cdot 0.7\ (= 0.127575)$$

$$S_{22} \approx 0.45^2 \cdot (2 \cdot 0.45)^2 \cdot 0.7\ (= 0.1148175)$$

etc. (It is recommended to stop at $S_{11} \approx 0.45 \cdot 1^2 \cdot 0.7$ etc)

If the calculation is persued it becomes:

$$\left(\begin{bmatrix} 0.14175 & 0.127575 \\ 0.127575 & 0.1148175 \end{bmatrix} - \begin{bmatrix} 0.08575 & 0.0471625 \\ 0.0471625 & 0.025939375 \end{bmatrix}\right) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0.49 \\ 0.441 \end{bmatrix} + \begin{bmatrix} 0.0496125 \\ 0.027286875 \end{bmatrix}$$

giving $\alpha_1 \approx -6.9199$ and $\alpha_2 \approx 11.5296$.

2. The weak formulations is of course the same as above. Two second order elements: Use quadratic base functions. On element 1 we have $x_1 = 0.1$, $x_2 = 0.3$, and $x_3 = 0.5$.

$$\varphi_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \qquad \varphi_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}, \qquad \varphi_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

with S, Q, B and F as in exercise 1 (except that indices goes from 1 to 3 and the intgeration is from 0.1 to 0.3. The derivatives are

$$\varphi_1' = \frac{(2x - x_2 - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \qquad \varphi_2' = \frac{(2x - x_1 - x_3)}{(x_2 - x_1)(x_2 - x_3)}, \qquad \varphi_3' = \frac{(2x - x_1 - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Using midpoint quadrature (it was said to be OK!) we have $\hat{x} = x_2 = (x_1 + x_3)/2$ and $\varphi_1(\hat{x}) = \varphi_3(\hat{x}) = \varphi_2'(\hat{x}) = 0$ thus

$$S = \begin{pmatrix} \alpha_1 & 0 & -\alpha_1 \\ 0 & 0 & 0 \\ -\alpha_1 & 0 & \alpha_1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ \gamma_1 \\ 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $\varphi_1'(\hat{x}) = -1/L_1 = -\varphi_3'(\hat{x})$ and $\varphi_2(\hat{x}) = 1$ giving $\alpha_1 = -\hat{x}^2/L_1$ and $\beta_1 = \hat{x}^2 \cdot 1^2 \cdot L_1$ and $\gamma_1 = \hat{x}^2 \cdot 1 \cdot L_1$, where $L_1 = x_3 - x_1$.

The same thing holds for element two (now $\hat{x} = x_4 = (x_5 + x_3)/2$),

$$S = \begin{pmatrix} \alpha_2 & 0 & -\alpha_2 \\ 0 & 0 & 0 \\ -\alpha_2 & 0 & \alpha_2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ \gamma_2 \\ 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 2.56 \end{pmatrix}$$

giving the global matrices after assembling

$$\begin{pmatrix} \alpha_1 & 0 & -\alpha_1 & 0 & 0 \\ 0 & -\beta_1 & 0 & 0 & 0 \\ -\alpha_1 & 0 & \alpha_1 + \alpha_2 & 0 & -\alpha_2 \\ 0 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & -\alpha_2 & 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma_1 \\ 0 \\ \gamma_2 \\ 2.56 \end{pmatrix}$$

Finally adjust for Dirichlet BC

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\beta_1 & 0 & 0 & 0 \\ -\alpha_1 & 0 & \alpha_1 + \alpha_2 & 0 & -\alpha_2 \\ 0 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & -\alpha_2 & 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 1 \\ \gamma_1 \\ 0 \\ \gamma_2 \\ 2.56 \end{pmatrix}$$

3. The DE $-\nabla \cdot (x\nabla u) = 2$ has weak formulation

$$\int_\Omega -\nabla \cdot (x\nabla u) v dx = \int_\Gamma (x\nabla u) v \cdot \hat{n} ds \int_\Omega (x\nabla u)\nabla v dx == \int_\Omega 2v dx$$

where $\int_\Gamma = 0$ since we have zero Neumann BC:s. Using the standard triangle element

$$\phi_1 = 1 - \xi - \eta, \qquad \phi_2 = \xi, \qquad \phi_3 = \eta$$

we have

$$B = \begin{bmatrix} \frac{\partial \phi_1}{\partial \xi} & \frac{\partial \phi_2}{\partial \xi} & \frac{\partial \phi_3}{\partial \xi} \\ \frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \frac{\partial \phi_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The element stiffness matrix will be

$$S_k = \int_0^1 \int_0^\eta x(\xi,\eta)(J^{-1}B)^T(J^{-1}B)\det J \;\, d\xi d\eta = (J^{-1}B)^T(J^{-1}B)\det J \;\; \int_0^1 \int_0^\eta x(\xi,\eta)\; d\xi d\eta$$

since the Jacobian is a constant over each element (because of linear basis functions). The components of the element load vector will be (with 1-point quadrature)

$$f_i = \int_0^1 2 \cdot \phi_i(\xi,\eta) \; \det J \;\, d\xi d\eta \approx 2 \cdot \frac{1}{3} \det J \cdot \frac{1}{2} = \frac{1}{3} \det J$$

The Jacobian is obtained by the iso-parametric mapping $\bar{x}(\xi,\eta) = \sum_{i=1}^3 \phi_i(\xi,\eta) \cdot \bar{x}_i$, here:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= (1-\xi-\eta)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \xi \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \eta \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \xi \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} + \eta \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \end{aligned}$$

The Jacobian is thus

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$

I now number the nodes, anticlockwise, starting with $(2,1)$ as number 1 and call the lower (right-most) triangle element 1.
On element 1 the nodes are 1, 2 and 3. The Jacobian is thus

$$J = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \qquad \text{with} \qquad \det J = 3$$

Using 1-point quadrature (at $\xi = \eta = 1/3$) we have $\hat{x} = (x_1 + x_2 + x_3)/3 = 8/3$ and $\int_0^1 \int_0^\eta x(\xi,\eta)\; d\xi d\eta \approx 8/3 \cdot 1/2 = 4/3$ XXX giving the local stiffness matris and element vector

$$S^{(1)} = \frac{1}{3}\begin{bmatrix} 9 & -9 & 0 \\ -9 & 10 & -1 \\ 0 & -1 & 1 \end{bmatrix} \frac{4}{3} \qquad \text{and} \qquad F^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

On element 2 the nodes are 1, 3 and 4. The Jacobian is thus

$$J = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} \qquad \text{with} \qquad \det J = 3$$

and $(x_1 + x_3 + x_4)/3 = 7/3$ giving the local stiffness matris and element vector

$$S^{(2)} = \frac{1}{3}\begin{bmatrix} 1 & -0 & -1 \\ 0 & 9 & -9 \\ -1 & -9 & 10 \end{bmatrix} \cdot \frac{7}{3} \cdot \frac{1}{2} \qquad \text{and} \qquad F^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The global matrices are thus

$$S = \begin{pmatrix} S_{11}^{(1)} & S_{12}^{(1)} & S_{13}^{(1)} & 0 \\ S_{21}^{(1)} & S_{22}^{(1)} & S_{23}^{(1)} & 0 \\ S_{31}^{(1)} & S_{32}^{(1)} & S_{33}^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} S_{11}^{(2)} & 0 & S_{12}^{(2)} & S_{13}^{(2)} \\ 0 & 0 & 0 & 0 \\ S_{21}^{(2)} & 0 & S_{22}^{(2)} & S_{23}^{(2)} \\ S_{31}^{(2)} & 0 & S_{32}^{(2)} & S_{33}^{(2)} \end{pmatrix} = \begin{pmatrix} S_{11}^{(1)} + S_{11}^{(2)} & S_{12}^{(1)} & S_{13}^{(1)} + S_{12}^{(2)} & S_{13}^{(2)} \\ S_{21}^{(1)} & S_{22}^{(1)} & S_{23}^{(1)} & 0 \\ S_{31}^{(1)} + S_{21}^{(2)} & S_{32}^{(1)} & S_{33}^{(1)} + S_{22}^{(2)} & S_{23}^{(2)} \\ S_{31}^{(2)} & 0 & S_{32}^{(2)} & S_{33}^{(2)} \end{pmatrix}$$

$$F = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

Adjust for the known Dirichlet boundary conditions in nodes 1 and 2 ($u_1 = 2^2 = 4$ and $u_2 = 3^2 = 9$)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ S_{31}^{(1)} + S_{21}^{(2)} & S_{32}^{(1)} & S_{33}^{(1)} + S_{22}^{(2)} & S_{23}^{(2)} \\ S_{31}^{(2)} & 0 & S_{32}^{(2)} & S_{33}^{(2)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 2 \\ 1 \end{pmatrix}$$

4. For the weak formulation we will use Gauss' Theorem which is

$$\int_\Omega \nabla \cdot \bar{w} \; d\Omega = \int_\Gamma \bar{w} \cdot \hat{n} \; dS$$

where Γ is the close surface surrounding Ω. From the rule of derivation of products we obtain

$$\nabla \cdot (\alpha \bar{w}) = \alpha (\nabla \cdot \bar{w}) + \bar{w} \cdot (\nabla \alpha)$$

We thus have

$$\int_\Omega \nabla \cdot (\bar{q} \; v) \; d\Omega = \int_\Omega (\nabla \cdot \bar{q}) \, v \; d\Omega + \int_\Omega \bar{q} \cdot \nabla v \; d\Omega \;\Rightarrow\; \int_\Omega (\nabla \cdot \bar{q}) \, v \; d\Omega = \int_\Gamma \hat{n} \cdot \bar{q} \; v \; ds - \int_\Omega \bar{q} \cdot \nabla v \; d\Omega$$

The differential equation read

$$-\nabla (k \; \nabla u) + \gamma \; u = f \qquad \text{on } \Omega$$

a weak formulation then is

$$\int_\Omega (-\nabla (k \; \nabla u)) \;\; v \; d\Omega + \int_\Omega \gamma \; u \; v \; d\Omega = \int_\Omega f \; v \; d\Omega$$

The first integral is changed using Gauss' Theorem with $\bar{q} = -k \nabla u$:

$$\int_\Omega (-\nabla (k \; \nabla u)) \;\; v \; d\Omega = \int_\Gamma \hat{n} \cdot (-k \; \nabla u) \;\; v \; ds - \int_\Omega (-k \; \nabla u) \cdot \nabla v \; d\Omega$$

The boundary Γ is split in three parts, Γ_1 where $y = 0$, Γ_2 where $x = 0$ and the curved boundary Γ_3. On Γ_1 and Γ_3 we have Dirichlet boundary conditions, thus we have $v = 0$ there. On Γ_2 we have a non-zero Neumann boundary conditions, thus this is the only part of $\int_\Gamma$ which remains.

$$\int_\Gamma \hat{n} \cdot (-k \; \nabla u) \;\; v \; ds = \int_{\Gamma_1} (+1) \; v \; ds$$

This leads to the following weak formulation: Find u such that

$$\int_\Omega k \; \nabla u \cdot \nabla v \; d\Omega + \int_\Omega \gamma \; u \; v \; d\Omega = \int_\Omega f \; v \; d\Omega - \int_{\Gamma_1} v \; ds$$

for all v such that $v = 0$ on Γ_1 and Γ_3 (and u and v must be continuous and once differentiable).

5.

Not complete answer!!! Only hints:

- Standard refinement of quad: put new nodes at midsides and center of element. Connect with lines. (Splits each quad into four quads)
- Standard refinement of triangle: put new nodes at midsides of element. Connect new nodes with lines. (Splits each triangle into four triangles)
- The Delauney criteria is the Delauney circle criteria. (in short: the circle circumventing the triangular element should not contain any other node.)