Examination paper Finite Elements, DN2260

14-19, October 25, 2007

Read *all* the questions before starting work. Do the easy ones first. Check carefully that your initially derived equations are correct. Ask if you are uncertain. Answers MUST be motivated. Paginate and write your name on EVERY page handed in. Answers in English or Swedish, please. Open books: ***only*** the course book is allowed.

A total of 20 out of max 50 points guarantees a "pass" (3, ECTS E). The results will be e-mailed to participants by Nov. 5, 2007. Papers are kept at the CSC Student Office for a year and then destroyed. Complaints to J.Oppelstrup by December 1, 2007, after which the results are irrevocable. The next examination paper will be given in December, 2007 and/or January, 2008.

P1. (10)

Consider the boundary value problem

$$-(2+x)u_{xx} - u_x + u = f(x)$$
$$u(0) = 0, u_x(1) = 0$$

(4) a) Give the bilinear form $a(.\,,.)$ and the linear form $L(.)$ in the variational formulation: Find u in V such that $a(u,v) = L(v), \forall v \in V$ and describe the space V. Hint: the equation can be written $-(\alpha(x)u_x)_x + \beta(x)u = f$. What are α and β?

(4) b) Show that the energy and Sobolev norms,

$$\|v\|_E = \sqrt{a(v,v)}, \|v\|_{H^1}^2 = \int_0^1 \left(v^2 + v_x^2\right)dx$$

are equivalent, i.e. there are numbers k_1 and k_2 such that for all v,

$$k_1\|v\|_E \le \|v\|_{H^1} \le k_2\|v\|_E$$

(2) c) Show that the equation is V-elliptic. What is the value of the constant in the defining inequality?

P2. (10)

(3) a) Describe the assembly process for computing the <u>*stiffness matrix*</u> from <u>*element matrices*</u> and the <u>*load vector*</u> of the linear system from the finite element discretization of a boundary value problem. Define the underlined quantities (don't compute any integrals) taking

$$-(a(x)u_x)_x = f(x), a > 0$$
$$u(0) = 1, a(1)u_x(1) = 0$$

as example.

(3) b) Consider the boundary value problem

$$u_x^\varepsilon - \varepsilon u_{xx}^\varepsilon = 1, \varepsilon \ge 0 \quad on\,[0,1]$$
$$u(0) = u(1) = 0$$

Sketch the Galerkin cG(1) solution for elements of size h with ε << h. Explain the "artificial diffusion" technique to overcome the problem.

(4) c) Consider the initial value problem for the parabolic system (u in $\mathbf{R}^n$)

$$u_t + Au = b$$
$$u(0) = 0$$

It holds that $\mathbf{v}^T\mathbf{A}\mathbf{v} \ge \delta\mathbf{v}^T\mathbf{v} = \delta\|\mathbf{v}\|_2^2, \delta > 0$. Discretize by the implicit method on the grid $\{t_n\}$, with time-steps $t_n - t_{n-1} = k_n, t_0=0$

$$u_n - u_{n-1} = k_n\left(b - Au_n\right)$$
$$u_0 = 0$$

i) Show that $\|\mathbf{w}_n\|_2 \le \frac{1}{1+\delta k_n}\|\mathbf{w}_{n-1}\|_2$ (1) where $\mathbf{u}_n = \mathbf{A}^{-1}\mathbf{b} + \mathbf{w}_n$

ii) Show that if $\delta k_n < 1$, $\frac{1}{1+\delta k_n} < e^{-\delta k_n/2}$ (2) and $\|\mathbf{w}_n\|_2 \le e^{-\frac{\delta t_n}{2}}\left\|\mathbf{A}^{-1}\mathbf{b}\right\|_2$ (you can use (1) and (2) even if you did not prove them) (Note: the inequality (2) is *not* sharp!)

P3. (10)

(4) a) Consider the boundary value problem

$$Lu = -\Delta u + xu_x = f(x,y) \text{ in } \Omega$$
$$u = 1 \text{ on } \partial\Omega$$

What is the associated bilinear form? Is it symmetric? What is the adjoint operator L^* (remember the boundary conditions!) Is L self-adjoint?

(4) b) The abstract problem $Lu = f$, u in V, is a boundary value problem on Ω (the one in a), for example), and its Galerkin approximation U in V_h, with error $e = u - U$. Consider the quantity $E = \int_\Omega w(x)e(x)dx$

The residual is defined by $Le = R(U)$. Show that $E = -\int_\Omega R(U)\varphi dx$ (*) when φ is chosen to solve a particular dual problem. What? Hint:

$$E = \int_\Omega [we + \varphi(Le - R(U))]dx \text{ and } \int_\Omega \varphi Le dx = \int_\Omega eL^*\varphi dx$$

(2) c) (You can use (*) even if you did not prove it). Now suppose w is a thin pulse, non-zero only in a small neighborhood of $x = x_0$. Explain how to use (*) to choose which elements to refine when the error at x_0 should be controlled.

P4. (10)

Consider the initial-boundary value convection-diffusion problem ($k > 0$)

$$u_t + \beta\cdot\nabla u - k\Delta u = 0 \ (\mathbf{x},t)\in\Omega\times[0,T], \mathbf{x} = (x,y)$$
$$u(\mathbf{x},t) = 0, \mathbf{x}\in\partial\Omega, u(\mathbf{x},0) = u_0(\mathbf{x})$$

with Ω the unit disk $|\mathbf{x}|<1$ and velocity field $\beta = (1,2)$ (i.e, constant, translatory motion).

(5) a) Prove the stability estimate:

$$\|u(.,T)\|_2^2 + 2k\int_0^T\|\nabla u\|_2^2 dt = \|u_0\|_2^2$$

Hint: Multiply by u, integrate over Ω, and use $\nabla\cdot\beta = 0$ so that

$$u\beta\cdot\nabla u = \frac{1}{2}(\beta\cdot\nabla)u^2 = \frac{1}{2}\nabla\cdot(\beta u^2)$$

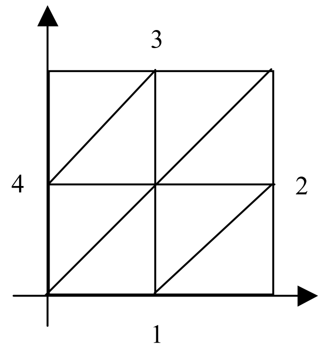
and then apply the divergence theorem.

(5) b) For $\beta = 0$, derive the estimate for the gradients,

$$\|\nabla u(.,T)\|_2 \le \|\nabla u_0\|_2$$

Hint: Multiply by Δu, integrate, use integration by parts (fill in the details): $\int_\Omega \Delta u \cdot u_t dx = -\int_\Omega \nabla u \cdot \nabla u_t dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega \|\nabla u\|^2 dx$

P5. (10)



The cG(1) Galerkin scheme is applied with the standard piecewise linear basis functions on right-angled triangles to the Poisson problem (see sketch)

$$-\Delta u = 1 \text{ in } [0,1]^2$$
$$u = 1 \text{ on } 3,4 \text{ and } 1$$
$$u_x + u = 0 \text{ on } 2$$

(2) a) Set $u = v + f(x,y)$ and choose f such that the Dirichlet boundary conditions of the problem for v become homogeneous.

(3) b) Write the variational formulation for the v-problem, give the bi-linear form and the linear form.

(3) c) Compute the integrals that appear in the Galerkin solution, both the system matrix and the load vector. Don't forget the contributions from the boundary.

(2) d) and solve for the two unknowns.