Extra lectures, etc.
Mon Oct 15 9(.15) -- 11 K53 (chemistry building)
Lecture: summing up: convection-diffusion; review
Tue Oct 16 9(15) -- 11 E33
Problem solving class: Old exam paper; review
Thu Oct 18 15 - -- E33
Questions & answers. Old exam paper.

Remember:
1. Deadline **Fri Oct 19** for Project B and Problem set B (i.e., mailbox will be emptied some time Mon Oct 22)
2. Completions of Project A and Problem set A will be accepted if handed in by the same deadline
3. Hint for those who want to improve their mesh refinement algorithm so the angles do not deteriorate with repeated refinement:
from:
Comparison of Triangular Adaptive Mesh Refinement Methods
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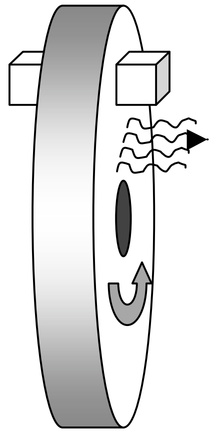
… Randolph Bank’s algorithm for regular refinement. The algorithm requires book-keeping so triangles that are bisected in one step can be re-combined and possibly split differently in the next step;
Triangles with only one nonconforming edge (i.e, with a hanging node) are bisected at the end of the process and previously bisected triangles are merged before refinement.
Thus, the smallest angle obtained is bounded by one half the smallest angle in M0. It may happen that *the resulting refined mesh M_i, may not have M_{i-1} nested within it.*)* If M_{i-1} is nested within Mi interpolation, etc., is simple.
*) Give an example of M_i not being nested in M_{i-1}

REGULAR-REFINEMENT(T)
Merge all bisected triangles *)
Mark all triangles that have an unacceptable error and add them to T0
i = 0
WHILE Ti is not empty **DO**
    Regularly refine marked triangles in T_i
    Let T_i be the set of triangle with two or more nonconforming edges
    IF T_i is empty **THEN**
        i = i + 1
        Mark all triangles that have an unacceptable error and add them to T_i
    END IF
END WHILE
Bisect remaining nonconforming triangles across their nonconforming edge.

Suggestion for "application project" (Proj B:4)



Compute the temperature distribution in a disc brake.

Two approaches
A. Steady, fixed coordinate system: The Euler model
B. Periodic in time, moving coordinate system: The Lagrange model

The disc spins at ω rad/sec, has inner diameter *Ri* and outer diameter *Ro*.
It is heated by friction in the caliper area, cooled by convection and radiation .

A 2D somewhat idealized model (list a few of the simplifications you note) for the temperature *u*(x,y,t) becomes

$$d\rho C(\frac{\partial u}{\partial t} + \beta(x,y)\cdot\nabla u) = d\nabla\cdot\left(k\nabla u\right) - hu + Q(x,y,t)$$
$$k\nabla u\cdot\mathbf{n} = 0 \text{ on boundaries } r = Ri\ ,\ r\ =\ Ro \qquad (1)$$
$$u(x,y,0)\ =\ 0, \text{ velocity field } \beta(x,y) = \omega(\text{-}y,x)$$

on Ω, a circular ring with inner radius *Ri* and outer radius *Ro*. *d* is the half-thickness (0.005m), *Q(x,y)* (W/m^2) the distributed friction heating, *h* (100 W/m^2/s) the convective heat transfer coefficient, the ambient temperature is 0, ρ density (7800 kg/m^3), *c* specific heat, and *k* heat conductivity (400 W/m/K). Let the caliper be an *a* x *b* rectangle *c* m from the center.

1. Show that by proper choice of scale factors ($L = Ro$) for length, temperature (you choose!) , and time ($T = 1/\omega$) , the model becomes

$$\frac{\partial u}{\partial t} + \beta(x,y)\cdot\nabla u - \varepsilon\Delta u = a(\chi(x,y) - u)$$
$$\nabla u\cdot\mathbf{n} = 0 \text{ on boundaries } r = Ri/Ro\ ,\ r\ =\ 1$$
$$u(x,y,0)\ =\ 0, \text{ velocity field } \beta(x,y) = (\text{-}y,x) \qquad \text{(EUL)}$$
$$\varepsilon = \frac{k}{\omega\rho cd\cdot Ro^2}, a = \frac{Q}{\omega\rho cd}, \chi \text{ is 1 in caliper area, 0 elsewhere}$$

2. For the steady solution (d/d*t* = 0), what is the arithmetic mean temperature $u_m = \frac{1}{|\Omega|}\int_\Omega u dxdy$? Hint: Integrate the equation over Ω and use the divergence theorem and that div$\beta = 0$, i.e. $\beta\cdot\nabla u = \nabla\cdot\left(\beta u\right)$.

A. Euler model

Implement the steady (d/d*t* = 0) equation (EUL) in Puffin. Play with the parameters ε and *a* to understand the influence on the solution.
1. Send *a* to ∞. What happens? … and *a* –> 0 ? *k* –> ∞? Why is *k* –>0 not reasonable ?
Hints:
1. We may define the characteristic function of the caliper area (1 inside, 0 outside) in matlab like

```
Q=((x>c)&(x<a+c)&(abs(y)<b/2))*1;
```

2. Use the `square` mesh on [0,1] x [0,1] (and refinements thereof) by mapping it to the circular ring: Let the meshpoints from `square` be (ξ_i,η_i) , then

$$(x_i, y_i) = (r_i\cos\phi_i, r_i\sin\phi_i), r_i = R_i + (R_o - R_i)\xi_i, \phi_i = 2\pi\eta_i$$

are gridpoints on the circular ring. There are duplicates of the points $\phi = 0$ and 2π. Fix that by
1) changing node numbers which point to the duplicates to point to the originals,
2) then removing the duplicates.

B Lagrange model (more complicated!)

In an (X,Y) coordinate system which rotates with the disc

$$(X,Y) = (x\cos t - y\sin t, x\sin t + y\cos t)$$

there is no convective heat transfer in the disc:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \varepsilon\Delta_X u = a(\chi(x(t), y(t)) - u) \\
&\nabla_X u \cdot \mathbf{N} = 0 \text{ on boundaries } r = Ri/Ro\ ,\ r = 1 \qquad \text{(LAG)} \\
&u(X,Y,0) = 0, \\
&(x,y) = (r\cos(\phi + t), r\sin(\phi + t)), r = \sqrt{X^2 + Y^2}, \tan\phi = Y/X
\end{aligned}$$

So the equation is simpler, but with the convection gone, instead the source becomes 2π-periodic in time:

$$\chi = \sum_{m=-\infty}^{\infty} f_m(X,Y)e^{imt}$$

It follows, that after long time u tends to a function also 2π-periodic in time:

$$u(X,Y,t) = \sum_{m=-\infty}^{\infty} u_m(X,Y)e^{imt}$$

Will Puffin do complex numbers and solve this correctly? Try!

To simplify matters, we change the caliper area to a circular sector: $c < r < a+c$, $-\phi_b<\phi<+\phi_b$.

Now we have

$$\chi = \gamma(r)g(\phi + t), \gamma(r) = \begin{cases} 1, c < r < c + a \\ 0, elsewhere \end{cases}, g(\phi) = \begin{cases} 1, -\phi_b < \phi < +\phi_b \\ 0, elsewhere \end{cases}$$

$$g(x) = \sum_{m=-\infty}^{\infty} g_m e^{imx}, g_m = \frac{\sin m\phi_b}{\pi m}$$

and we derive the equations

$$\begin{aligned}
&imu_m - \varepsilon\Delta_X u_m = a(\gamma(r)g_m e^{im\phi} - u_m) \text{ in } \Omega \qquad \text{(LAG-FOUR)} \\
&\nabla_X u_m \cdot \mathbf{N} = 0 \text{ on } \partial\Omega
\end{aligned}$$

for the terms u_m, $m = -\infty, ..., -1, 0, 1, ..., +\infty$

The χ-function is discontinous and its Fourier series converges slowly: The f_m decay only like $1/m$. The u-series converges as slowly.

1. Check the formula for g_m and that you agree on the result of the variable transformation leading to (LAG) and (LAG-FOUR).

2. Take 3, 5, 11 and 21 (m = -10,-9,…,-1,0,1,…,9,10) terms in the series. Thus, the mesh should resolve $\cos 10\phi$ which means say 60 points in the angular direction and as many radially: γ is a pulse, too. You will observe in the solution smooth wiggles around the edges of the caliper. Look up Gibb's phenomenon in your Fourier analysis book and comment.