Example 2

$$-u'' = f \qquad 1 < x < 2$$

Solve the problem with 1D FEM and tent functions and discretization points $x_1 = 1$, $x_2 = 1.5$, $x_3 = 1.7$ and $x_4 = 2$

a) Calculate the stiffness matrix and load vector assuming $f = 1$ and boundary conditions $u'(1) = u'(2) = 0$.

b) The system obtained in a cannot be solved, the stiffness matrix is singular. Why?

c) Solve assuming $f = 1$ and boundary conditions $u(1) = 5$ and $u(2) = 7$.

d) Solve assuming $f = x$ and boundary conditions $u(1) = 5$ and $u(2) = 7$.

********* Short answer **********

a) A weak formulation is

$$\int_1^2 u'\, v'\, dx = \int_1^2 f\, v\, dx$$

i.e.

$$a(u, v) = \int_1^2 u'\, v'\, dx \qquad L(v) = \int_1^2 f\, v\, dx$$

The FEM system of equations then become (from the ansatz $U = \sum_i c_i\, \phi_i(x)$):

$$a(U, \phi_i) = L(\phi_i), \qquad i = 1..4$$

leading to

$$\mathbf{S}\,\mathbf{c} = \mathbf{f}$$

where

$$\mathbf{S}_{ij} = a(\phi_i, \phi_j) \quad \text{and} \quad \mathbf{f}_i = L(\phi_i) \quad \text{for } \ i = 1..4$$

Calculation is done elementwise. Start with number 1:

$$\mathbf{S}^{(1)}_{ij} = \int_{x_1}^{x_2} \phi_i'\, \phi_j'\, dx$$

Basis functions are

$$\phi_1^{(1)} = \frac{x_2 - x}{x_2 - x_1} = \frac{x_2 - x}{L_1} \quad \Longrightarrow \quad \phi_1' = \frac{-1}{L_1}$$

$$\phi_2^{(1)} = \frac{x - x_1}{x_2 - x_1} = \frac{x - x_1}{L_1} \quad \Longrightarrow \quad \phi_2' = \frac{1}{L_1}$$

giving

$$\mathbf{S}^{(1)}_{11} = \int_{x_1}^{x_2} (\phi_1')^2\, dx = \int_{x_1}^{x_2} \left(\frac{-1}{L_1}\right)^2 dx = \frac{1}{L_1}$$

$$\mathbf{S}_{12}^{(1)} = \int_{x_1}^{x_2} \phi_1' \, \phi_2' \, dx = \int_{x_1}^{x_2} \left(\frac{-1}{L_1}\right)\left(\frac{1}{L_1}\right) dx = \frac{-1}{L_1} = \mathbf{S}_{21}^{(1)}$$

$$\mathbf{S}_{22}^{(1)} = \int_{x_1}^{x_2} (\phi_2')^2 \; dx = \int_{x_1}^{x_2} \left(\frac{1}{L_1}\right)^2 dx = \frac{1}{L_1}$$

Thus the element matrix is

$$\mathbf{S}^{(1)} = \frac{1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element load vector elements are

$$\mathbf{f}_1^{(1)} = \int_{x_1}^{x_2} 1 \cdot \phi_1 \; dx = \int_{x_1}^{x_2} \frac{x_2 - x}{L_1} \, dx = \frac{L_1}{2}$$

$$\mathbf{f}_2^{(1)} = \int_{x_1}^{x_2} 1 \cdot \phi_2 \; dx = \int_{x_1}^{x_2} \frac{x - x_1}{L_1} \, dx = \frac{L_1}{2}$$

Thus the element load vector is

$$\mathbf{f}^{(1)} = \frac{L_1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From this we conclude that for element k we have

$$\mathbf{S}^{(k)} = \frac{1}{L_k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{(k)} = \frac{L_k}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

giving

$$\mathbf{S}^{(1)} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \; \mathbf{S}^{(2)} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{S}^{(3)} = \begin{bmatrix} 10/3 & -10/3 \\ -10/3 & 10/3 \end{bmatrix}$$

and

$$\mathbf{f}^{(1)} = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}, \; \mathbf{f}^{(2)} = \begin{bmatrix} 0.10 \\ 0.10 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{(3)} = \begin{bmatrix} 0.15 \\ 0.15 \end{bmatrix}$$

giving the global matrices

$$\mathbf{S} = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2+5 & -5 & 0 \\ 0 & -5 & 5+10/3 & -10/3 \\ 0 & 0 & -10/3 & 10/3 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} 0.25 \\ 0.25+0.10 \\ 0.10+0.15 \\ 0.15 \end{bmatrix}$$

b) We then want to solve this system $\mathbf{Sc} = \mathbf{f}$. However, we run into trouble: the matrix $\mathbf{S}$ is singular! How come? Well, the reason is that we try to solve an impossible problem! $u'' = -1$ can be solved analytically: $u' = -x + C_1$ and $u = -x^2/2 + C_1 x + C_2$. The integration constants C_1 and C_2 should be determined by the boundary conditions. But the conditions are both on u'; $u'(1) = -1 + C_1 = 0$ and $u'(2) = -2 + C_1 = 0$ giving impossible demands on C_1 and none on C_2. Alternative reasoning: no second degree polynomial can have $u' = 0$ at two different places. Thus no solution exists with these boundary conditions! We thus change the boundary conditions in the next exercise.

c) Now the boundary conditions are $u(1) = 5$ and $u(2) = 7$ which lead to the same weak form as in exercise a. (but with the added condition that test functions should be zero at $x = 1$ and $x = 2$. This is no problem except for ϕ_1 and ϕ_4 which we deal with now).

When u is known at a point we just replace that line in the system of equations with a 1 on the diagonal in $\mathbf{S}$ and the value of u in the load vector. Thus here

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 2+5 & -5 & 0 \\ 0 & -5 & 5+10/3 & -10/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} 5 \\ 0.25+0.10 \\ 0.10+0.15 \\ 7 \end{bmatrix}$$

giving

$$\mathbf{c} = [5 \quad 6.125 \quad 6.505 \quad 7]^T$$

d) Compared to exercise c we have changed f, nothing else. This means that the stiffness matrix $\mathbf{S}$ is unchanged. However, we can no longer give an easy form for $\mathbf{f}^{(k)}$ They have to be calculated properly:

$$\mathbf{f}_1^{(k)} = \int_{x_k}^{x_{k+1}} x \cdot \phi_1^{(k)} \, dx = \int_{x_k}^{x_{k+1}} x \; \frac{x_{k+1} - x}{L_k} \, dx$$

and

$$\mathbf{f}_2^{(k)} = \int_{x_k}^{x_{k+1}} x \cdot \phi_2^{(k)} \, dx = \int_{x_k}^{x_{k+1}} x \; \frac{x - x_k}{L_k} \, dx$$

giving

$$\mathbf{f}^{(1)} = \begin{bmatrix} 0.2917 \\ 0.3333 \end{bmatrix}, \quad \mathbf{f}^{(2)} = \begin{bmatrix} 0.1567 \\ 0.1633 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{(3)} = \begin{bmatrix} 0.2700 \\ 0.2850 \end{bmatrix}$$

which assemblated is

$$\mathbf{f} = \begin{bmatrix} 0.2917 \\ 0.3333+0.1567 \\ 0.1633+0.2700 \\ 0.2850 \end{bmatrix}$$

and after adjustment for the essential boundary conditions

$$\mathbf{f} = \begin{bmatrix} 5 \\ 0.3333+0.1567 \\ 0.1633+0.2700 \\ 7 \end{bmatrix}$$

Solving $\mathbf{Sc} = \mathbf{f}$ we get

$$\mathbf{c} = [5 \quad 6.1875 \quad 6.5645 \quad 7]^T$$

NC 2002-11-05