DN2660 The Finite Element Method: Written Examination
Tuesday 2008-10-21, 14-19
Coordinator: Johan Jansson

Aids: none Time: 5 hours

Answers must be given in English. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be given zero points, while a good explanation with a wrong answer can give some points. Maximum is 30 points.

Good luck,
Johan

Problem 1 - Galerkin's method

Consider the equation:

$$-\nabla \cdot (a(x)\nabla u(x)) + b(x)u(x) = f(x), \quad x \in \Omega$$
$$u(x) = 0, \quad x \in \Gamma$$

where $a(x), b(x)$ and $f(x)$ are known coefficients.

Recall the formula:

$$\int_\Omega D_{x_i} vw dx = \int_\Gamma vwn_i ds - \int_\Omega v D_{x_i} w dx, \quad i = 1, 2, ..., d$$

1. (2p) Formulate a finite element method (Galerkin's method) for the equation using piecewise linear approximation (cG(1)).
2. (1p) Explain what the Galerkin orthogonality means, both in general and for this equation.
3. (1p) Change the boundary condition to homogenous Neumann, and show the effect on the FEM formulation.
4. (1p) Discuss using piecewise polynomial basis functions versus using global basis functions with regard to the resulting linear system and adaptivity.

Problem 2 - Stability

Consider the heat equation with zero source:

$$\begin{aligned} \dot{u} - \Delta u &= 0, \quad x \in \Omega, \quad t \in [0, T] \\ u(0, x) &= u_0(x) \\ u(t, x) &= 0, \quad x \in \Gamma \end{aligned}$$

1. (2p) Derive the stability estimate (hint: multiply by u):

$$\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2 dt = \|u(0)\|^2$$

   Explain what a stability estimate is in general, and give an interpretation what this particular stability estimate says about the temperature u.

2. (2p) Explain the basic concept behind a streamline diffusion stabilized finite element method.

Problem 3 - Assembly of a linear system

1. (3p) Formulate a general assembly algorithm of a linear system given a bilinear form $a(u, v)$ and linear form $L(v)$ representing a linear boundary value partial differential equation (PDE) in 2D/3D, with a piecewise linear Galerkin approximation (cG(1)). Include explanations of the following concepts:
   - Mesh
   - Map from reference cell
   - Formula for computation of a matrix and vector element
   - Quadrature

2. (2p) Define a basic linear boundary value PDE in 1D. Apply Galerkin's method, construct a simple mesh and compute a matrix element by hand (you don't have to use a general assembly algorithm here).

Problem 4 - Error estimation

Consider the equation:

$$\begin{aligned} -u'' + u &= f, \quad x \in [0, 1] \\ u(0) &= u(1) = 0 \end{aligned}$$

We define the energy norm for this equation: $\|w\|_E = \sqrt{a(w, w)} = \sqrt{\int_0^1 (w')^2 + w^2 dx}$

1. (2p) Show that Galerkin's method is optimal for the equation and derive an a priori error estimate in the energy norm $\|w\|_E$.

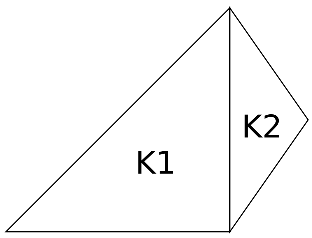
2. (2p) Derive an a posteriori error estimate in the energy norm (hint: you can use that $\|w\|_{L_2} \leq \|w\|_E$).

3. (2p) Sketch the basic steps for how to construct an error estimate of a general quantity of the error (e, ψ) using duality.

Problem 5 - Adaptivity

1. (3p) Formulate an adaptive finite element method based on an a posteriori error estimate with local mesh refinement given a tolerance TOL on a quantity or norm of the error $e = u - U$. Discuss why adaptivity is important.

2. (2p) Formulate the Rivara recursive bisection algorithm. Consider the mesh:



Mark the triangle K2 for refinement and perform the Rivara algorithm by hand, show all steps.

Problem 6 - Abstract formulation

1. (2p) Explain what the Lax-Milgram theorem says, what it requires to be satisfied, and what it can be used for.

2. (2p) Assume that we have, in an abstract formulation, a boundary value PDE: $a(u, v) = L(v), \quad \forall v \in V$

   We apply Galerkin's method and construct a discrete solution U which satisfies: $a(U, v) = L(v), \quad \forall v \in V_h$

   Show that the discrete solution U is optimal in the energy norm: $\|w\|_E = \sqrt{a(w, w)}$.

   We assume we can find a solution and that the energy norm exists.