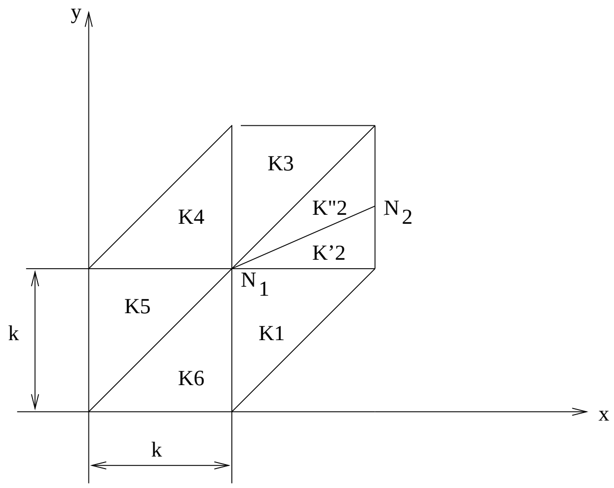
**TMA371/MAN660 Partial Differential Equations TM, IMP, E3, GU**
**2002-12-17. Solutions**

**1.**
Let $\Omega$ be the triangulated domain below. Compute the cG(1) solution of

$$\begin{cases} -\Delta u = 1, & \text{on } \Omega \\ u_x(1,y)=0, \ \ 1/2 \le y \le 1, & u(x,y)=0, \quad \text{on the rest of boundary.} \end{cases}$$




**Solution.** We split the boundary $\Gamma := \partial\Omega$ of the domain as $\Gamma = \Gamma_N + \Gamma_D$ with $\Gamma_N = \{(1,y) : 1/2 \le y \le 1\}$ and $\Gamma_D = \Gamma \setminus \Gamma_N$.
<u>Variational Formulation:</u> Using Green's formula we have that

$$\begin{aligned} \int_\Omega 1 \cdot v &= \int_\Omega -\Delta u v = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (\partial_n u) v \\ &= \{v = 0 \text{ on } \Gamma_D, \text{ and } \partial_n u = u_x = 0 \text{ on } \Gamma_N\} \\ &= \int_\Omega \nabla u \cdot \nabla v. \end{aligned} \tag{1}$$

Thus we have the <u>finite element formulation</u>: Find piecewise linear function $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega 1 \cdot v, \quad \forall v \in V_h. \tag{2}$$

**Remark.** It is natural to think $P_1 = (1/2, 1/2)$, $P_2 = (1, 1/2)$ and $P_3 = (1,1)$ as nodes. Then there are two possibilities:
(I) Normally we let the basis functions $\psi_2$, and $\psi_3$ (corresponding to the nodes $P_2$ and $P_3$, respectively) to have $K_1 \cap K_2$, and $K_2 \cap K_3$ as their supports, respectively. This however extends the Neumann boundary codition from $\Gamma_N$ to $\Gamma_N \cap \{(x,y) : y = x - 1/2, \ 1/2 \le x \le 1\} \cup \{(x,y) : y = 1, \ 1/2 \le x \le 1\}$, thus removing the Dirichlet condition from the line-segments $\{(x,y) : y = x - 1/2, \ 1/2 \le x \le 1\}$ and $\{(x,y) : y = 1, \ 1/2 \le x \le 1\}$.

(II) To circumvent this difficulty, we restrict the support of both basis functions $\psi_i,\ i=2,3$ to ONLY! $K_2$. And this choice introduce discontinuities and cannot be considered in cG(1).

Hence summing up, because of cG(1) approximation in this problem we cannot choose the boundary points $(1,1/2)$ and $(1,1)$ as nodes (this will either destroy the continuity or replace the Dirichlet condition on $\{(x,y) : 1/2 \le x \le 1\}$, with $0 \le y \le 1/2$ and $y=1$ by the Neumann condition. So the only adequarte way is the *refinement* of $K_2$ into two triangles $K_2'$ and $K_2''$, by letting, e.g. $N_2=(1,3/4)$, as in the figure. We define the test functions $\varphi_i(Nj)=\delta_{ij},\ i,j=1,2$. In order to have zero boundary condition on $\Gamma_D,\ \varphi_2$, has $K_2=K_2'\cap K_2''$ as its support. Let now

$$U(x,y)=U_1\varphi_1(x,y)+U_2\varphi_2(x,y).$$

where $U_i=U(N_i),\ i=1,2$. Observe that $\varphi_i$s are the bases functions for $V_h$ and thus the equation (2) is equivalent to the following system:

$$\int_\Omega \nabla\varphi_1\cdot\nabla\varphi_i U_1+\int_\Omega \nabla\varphi_2\cdot\nabla\varphi_i U_2+=\int_\Omega \varphi_i,\quad i=1,2. \tag{3}$$

Let now $\varphi_2'=\varphi|_{K_2'}$ and $\varphi_2''=\varphi|_{K_2''}$, then using a standard triangle $K_2$ with vertices $(0,0),\ (k,0)$, and $(k,k)$, for $\varphi_2'(x,y)=ax+by+c$ we have that

$$\begin{aligned}
&\varphi_2'(0,0)=0\Rightarrow c=0,\\
&\varphi_2'(k,0)=0\Rightarrow ak=0\Leftrightarrow a=0,\\
&\varphi_2'(k,k/2)=1\Rightarrow bk/2=1\Leftrightarrow b=2/k.
\end{aligned}$$

Thus $\varphi_2'(x,y)=\frac{2}{k}y$. Similarly for $\varphi_2''(x,y)=ax+by+c$ we have that

$$\begin{aligned}
&\varphi_2''(0,0)=0\Rightarrow c=0,\\
&\varphi_2''(k,k)=0\Rightarrow ak+bk=0\Leftrightarrow b=-a,\\
&\varphi_2''(k,k/2)=1\Rightarrow ak+bk/2=1\Leftrightarrow a=2/k,\ b=-2/k.
\end{aligned}$$

Thus $\varphi_2''(x,y)=\frac{2}{k}x-\frac{2}{k}y$.
Considering in standard triangles we can easily see from the figure that:

$$\begin{aligned}
&\nabla\varphi_1\Big|_{K_1}=\tfrac{1}{k}(1,1), &&\nabla\varphi_2\Big|_{K_1}=(0,0),\\
&\nabla\varphi_1\Big|_{K_2}=\tfrac{1}{k}(-1,0), &&\nabla\varphi_2\Big|_{K_2'}=\tfrac{1}{k}(0,2), \quad \nabla\varphi_2\Big|_{K_2''}=\tfrac{1}{k}(2,-2),\\
&\nabla\varphi_1\Big|_{K_3}=\tfrac{1}{k}(0,-1), &&\nabla\varphi_2\Big|_{K_3}=(0,0),\\
&\nabla\varphi_1\Big|_{K_4}=\tfrac{1}{k}(1,-1), &&\nabla\varphi_2\Big|_{K_4}=(0,0),\\
&\nabla\varphi_1\Big|_{K_5}=\tfrac{1}{k}(1,0), &&\nabla\varphi_2\Big|_{K_5}=(0,0),\\
&\nabla\varphi_1\Big|_{K_6}=\tfrac{1}{k}(0,1), &&\nabla\varphi_2\Big|_{K_6}=(0,0).
\end{aligned}$$

Now considering the intersections of the supports of the $\varphi$ functions we have that:

$$\begin{aligned}\int_\Omega \nabla\varphi_1\cdot\nabla\varphi_1 &= \sum_{j=1}^{6}\int_{K_j}\nabla\varphi_1\cdot\nabla\varphi_1\\ &= \frac{k^2}{2}\frac{1}{k^2}\Big\{(-1,1)\cdot(-1,1)+(-1,0)\cdot(-1,0)+(0,-1)\cdot(0,-1)\\ &\quad +(1,-1)\cdot(1,-1)+(1,0)\cdot(1,0)+(1,0)\cdot(1,0)\Big\}\\ &= \frac{1}{2}\{2+1+1+2+1+1\}=4.\end{aligned}$$

Note that $|K'|=|K''|=\frac{1}{2}k\cdot\frac{k}{2}=\frac{k^2}{4}$, thus

$$\begin{aligned}\int_\Omega \nabla\varphi_2\cdot\nabla\varphi_2 = \int_{K_2}\nabla\varphi_2\cdot\nabla\varphi_2 &= \int_{K_2'}\nabla\varphi_2'\cdot\nabla\varphi_2' + \int_{K_2''}\nabla\varphi_2''\cdot\nabla\varphi_2''\\ &= \frac{k^2}{4}\frac{1}{k^2}\Big\{(0,2)\cdot(0,2)+(2,-2)\cdot(2,-2)\Big\} = \frac{1}{4}\{4+8\}=3,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \nabla\varphi_1\cdot\nabla\varphi_2 = \int_{K_2}\nabla\varphi_1\cdot\nabla\varphi_2 &= \int_{K_2'}\nabla\varphi_1\cdot\nabla\varphi_2' + \int_{K_2''}\nabla\varphi_1\cdot\nabla\varphi_2''\\ &= \frac{k^2}{4}\frac{1}{k^2}\Big\{(-1,0)\cdot(0,2)+(-1,0)\cdot(2,-2)\Big\} = \frac{1}{4}\{-2\}=-1/2.\end{aligned}$$

As for the right hand side in (3) we have

$$\int_\Omega \varphi_1 = 6\cdot\frac{1}{3}\frac{k^2}{2}=k^2,\qquad \int_\Omega\varphi_2=\int_{K_2}\varphi_2=\cdots=1\cdot\frac{1}{3}\frac{k^2}{2}=k^2/6.$$

Thus recalling that $k=1/2$, we have the folowing system of equations:

$$\begin{pmatrix}4 & -1/2\\ -1/2 & 3\end{pmatrix}\begin{pmatrix}U_1\\ U_2\end{pmatrix}\begin{pmatrix}1/4\\ 1/24\end{pmatrix}\Rightarrow U_1=6U_2-1/12,\ \text{with}\quad U_2=\frac{7}{6(47)}.$$

**2.** See stability lemma in chapter 9 of the lecture notes.

**3.** Prove that if $u=0$ on the boundary of the unit square $\Omega$, then

$$\Big(\int_\Omega |u|^2\,dx\Big)^{1/2}\le\Big(\int_\Omega|\nabla u|^2\,dx\Big)^{1/2}.$$

**Solution.** We have that

$$\begin{aligned}|u(x)| = |u(x_1,x_2)-u(0,x_2)| &= \Big|\int_0^{x_1}\frac{\partial}{\partial x_1}u(\bar{x_1},x_2)\,d\bar{x_1}\Big|\\ &= \Big|\int_0^{x_1}1\cdot\frac{\partial}{\partial x_1}u(\bar{x_1},x_2)\,d\bar{x_1}\Big| \le \{\text{Cauchy's inequality}\}\\ &\le \Big(\int_0^{x_1}1^2\,d\bar{x_1}\Big)^{1/2}\cdot\Big(\int_0^{x_1}(\frac{\partial}{\partial x_1}u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)^{1/2}\\ &\le \Big(\int_0^{1}(\frac{\partial}{\partial x_1}u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)^{1/2}.\end{aligned}$$

This implies that

$$\begin{aligned}
\int_\Omega |u|^2\, dx &\le \int_\Omega \Big(\int_0^1 (\frac{\partial}{\partial x_1} u(\bar{x_1}, x_2))^2\, d\bar{x_1}\Big)\, dx \\
&= \int_0^1\int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial x_1} u(\bar{x_1}, x_2))^2\, d\bar{x_1}\Big)\, dx_1\, dx_2 \\
&= \int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial x_1} u(\bar{x_1}, x_2))^2\, d\bar{x_1}\Big)\, dx_2 = \int_0^1\int_0^1 (\frac{\partial}{\partial x_1} u(x_1, x_2))^2\, dx_1\, dx_2 \\
&= \int_\Omega (\frac{\partial}{\partial x_1} u(x_1, x_2))^2\, dx \le \int_\Omega |\nabla u|^2\, dx,
\end{aligned}$$

which gives the desired result:

$$\Big(\int_\Omega |u|^2\, dx\Big)^{1/2} \le \Big(\int_\Omega |\nabla u|^2\, dx\Big)^{1/2}.$$

**4.** Prove an a priori and an a posteriori error estimate (in the energy norm: $||u||_E^2 := ||u'||^2 + ||u||^2$) for the cG(1) finite element method for the problem

$$\begin{cases} -u'' + \alpha u' + u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

where $\alpha \ge 0$. For which value of $\alpha$ is the a priori error estimate optimal?

**Solution.** The Variational formulation:
Multiply the equation by $v \in V$, integrate by parts over $(0, 1)$ and use the boundary conditions to obtain

$$\text{(4)}\quad \text{Find } u \in V: \int_0^1 u'v'\, dx + \int_0^1 \alpha u' v\, dx + \int_0^1 uv\, dx = \int_0^1 fv\, dx, \quad \forall v \in V.$$

cG(1):

$$\text{(5)}\quad \text{Find } U \in V_h: \int_0^1 U'v'\, dx + \int_0^1 \alpha U' v\, dx + \int_0^1 Uv\, dx = \int_0^1 fv\, dx, \quad \forall v \in V_h.$$

From (1)-(2), we find The Galerkin orthogonality:

$$\text{(6)}\quad \int_0^1 \Big((u-U)'v' + \alpha(u-U)'v + (u-U)v\Big)\, dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v, w)_E = \int_0^1 (v'w' + vw)\, dx, \qquad \forall v, w \in V.$$

Note that

$$\text{(7)}\quad \int_0^1 e'e\, dx = \frac{1}{2}\int_0^1 \frac{d}{dx}\Big(e^2\Big)\, dx = \frac{1}{2}[e^2]_0^1 = 0.$$

Thus using (7) we have

$$\text{(8)}\quad ||e||_E^2 = \int_0^1 (e'e' + ee)\, dx = \int_0^1 (e'e' + \alpha e'e + ee)\, dx.$$

We split the second factor $e$ as $e = u - U = u - v + v - U$, with $v \in V_h$ and write

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 \Big( e'(u-U)' + \alpha e'(u-U) + e(u-U) \Big)\, dx = \Big\{ v \in V_h \Big\} \\
&= \int_0^1 \Big( e'(u-v)' + \alpha e'(u-v) + e(u-v) \Big)\, dx \\
&+ \int_0^1 \Big( e'(v-U)' + \alpha e'(v-U) + e(v-U) \Big)\, dx \\
&= \int_0^1 \Big( e'(u-v)' + \alpha e'(u-v) + e(u-v) \Big)\, dx,
\end{aligned} \tag{9}
$$

where, in the last step, we have used the Galerkin orthogonality to eliminate terms involving $U$. Now we can write

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 \Big( e'(u-v)' + e(u-v) + \alpha e'(u-v) \Big)\, dx \\
&\le ||e||_E \cdot ||u-v||_E + \alpha ||e'||_{L_2} ||u-v||_{L_2} \\
&\le ||e||_E \Big( ||u-v||_E + \alpha ||u-v||_{L_2} \Big) \le ||e||_E ||u-v||_E (1+\alpha),
\end{aligned} \tag{10}
$$

and derive the <u>a priori error estimate:</u>

$$
||e||_E \le ||u-v||_E (1+\alpha), \quad \forall v \in V_h.
$$

To obtain <u>a posteriori error estimates</u> the idea is to eliminate $u$-terms, by using the differential equation, and replacing their contributions by the data $f$. Then this $f$ combined with the remaining $U$-terms would yield to the residual error:
<u>A posteriori error estimate:</u>

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 (e'e' + ee)\, dx = \int_0^1 (e'e' + \alpha e'e + ee)\, dx \\
&= \int_0^1 (u'e' + \alpha u'e + ue)\, dx - \int_0^1 (U'e' + \alpha U'e + Ue)\, dx.
\end{aligned} \tag{11}
$$

Now using the variational formulation (4) we have that

$$
\int_0^1 (u'e' + \alpha u'e + ue)\, dx = \int_0^1 fe\, dx.
$$

Inserting in (11) and using (5) with $v = \Pi_k e$ we get

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 fe\, dx - \int_0^1 (U'e' + \alpha U'e + Ue)\, dx \\
&+ \int_0^1 (U'\Pi_h e' + \alpha U'\Pi_h e + U\Pi_h e)\, dx - \int_0^1 f\Pi_h e\, dx.
\end{aligned} \tag{12}
$$

Thus

$$\begin{aligned}
||e||_E^2 &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 \Big(U'(e-\Pi_h e)' + \alpha U'(e-\Pi_h e) + U(e-\Pi_h e)\Big)\,dx \\
&= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (\alpha U' + U)(e-\Pi_h e)\,dx - \sum_{j=1}^{M+1}\int_{I_j} U'(e-\Pi_h e)'\,dx \\
&= \{\text{partial integration}\} \\
&= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (\alpha U' + U)(e-\Pi_h e)\,dx + \sum_{j=1}^{M+1}\int_{I_j} U''(e-\Pi_h e)\,dx \\
&= \int_0^1 (f + U'' - \alpha U' - U)(e-\Pi_h e)\,dx = \int_0^1 R(U)(e-\Pi_h e)\,dx \\
&= \int_0^1 hR(U)h^{-1}(e-\Pi_h e)\,dx \le ||hR(U)||_{L_2}||h^{-1}(e-\Pi_h e)||_{L_2} \\
&\le C_i||hR(U)||_{L_2}\cdot||e'||_{L_2} \le ||hR(U)||_{L_2}\cdot||e||_E.
\end{aligned}$$

This gives the <u>a posteriori error estimate:</u>

$$||e||_E \le C_i||hR(U)||_{L_2},$$

with $R(U) = f + U'' - \alpha U' - U = f - \alpha U' - U$ on $(x_{i-1}, x_i)$, $i = 1, \dots, M+1$.

The a priori error estimate is optimal for $\alpha = 0$.

**5.** Consider the boundary value problem

$$\begin{cases} -\Delta u = 0, & \text{in a bounded domain } \Omega \subset \mathbb{R}^d,\ d = 2, 3. \\ \frac{\partial u}{\partial n} + u = g, & \text{on } \Gamma = \partial\Omega. \end{cases}$$

a) Prove the $L_2$ stability estimate

$$||\nabla u||_{L_2(\Omega)}^2 + \frac{1}{2}||u||_{L_2(\Gamma)}^2 \le \frac{1}{2}||g||_{L_2(\Gamma)}^2.$$

b) Verify the conditions on Riesz/Lax-Milgram theorems for this problem.

**Solution:** a) Using Greens formula we have that

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u\cdot\nabla u = -\int_\Omega (\Delta u)u + \int_{\partial\Omega}\frac{\partial u}{\partial n}u = \int_{\partial\Omega}(g-u)u.$$

In other words

$$||\nabla u||_{L_2(\Omega)}^2 + ||u||_{L_2(\Gamma)}^2 = \int_{\partial\Omega} gu \le ||g||_{L_2(\Gamma)}^2||u||_{L_2(\Gamma)}^2 \le \frac{1}{2}||g||_{L_2(\Gamma)}^2 + \frac{1}{2}||u||_{L_2(\Gamma)}^2,$$

which gives the desired estimate.

To show the Riesz/Lax-Milgram conditions we introduce the notation

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v + \int_{\partial\Omega} uv, \quad \text{and} \quad L(v) = \int_{\partial\Omega} gv.$$

Then $a(u,v)$ is a scalar product with the corresponding norm $||v||_a = a(v,v)^{1/2}$. For instance we have that $||v||_a = 0$, only if $v = 0$:

$$0 = ||v||_a^2 = a(u,v) = \int_\Omega |\nabla v|^2 + \int_{\partial\Omega} v^2 \ge \alpha\int_\Omega v^2, \qquad \text{for some } \alpha > 0 \Rightarrow v = 0.$$

Further $L(v)$ is bounded in this norm, e.g. if $||g||_{\partial\Omega} < \infty$, then

$$|L(v)| \leq ||g||_{\partial\Omega}||v||_{\partial\Omega} \leq ||g||_{\partial\Omega}||v||_a.$$

We can also apply Riesz theorem in the sense that there existes $u$ such that

$$a(u,v) = L(v), \qquad \forall v,$$

and $u$ is uniquely determined by

$$||u||_a = ||g||_{\partial\Omega}.$$

Moreover since

$$a(u,v) = -\int_\Omega \Delta uv + \int_{\partial\Omega} (\frac{\partial u}{\partial n} + u)v,$$

we have that

$$\Delta u = 0, \quad \text{in } \Omega \quad \frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma.$$

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