TMA371/MAN660 Partiella differentialekvationer TM, E3, GU

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

1. Show that the linear interpolant $\pi_1 f$ of a function f on an interval $I = [a, b]$ satisfies the interpolation estimates:

$$\text{(a)} \qquad ||f - \pi_1 f||_{L^\infty(a,b)} \leq (b-a)^2 ||f''||_{L^\infty(a,b)},$$

$$\text{(b)} \qquad ||f' - (\pi_1 f)'||_{L^\infty(a,b)} \leq (b-a) ||f''||_{L^\infty(a,b)}.$$

2. Let U be the $cG(1)$ approximation of u satisfying the initial value problem

$$\dot{u} + au = f, \quad t > 0, \qquad u(0) = u_0.$$

Let k be the time step and show that for $a = 1$,

$$|(u - U)(T)| \leq \min \Big(||k(f - \dot{U} - U)||_{L^\infty[0,T]}, T||k^2 \ddot{u}||_{L^\infty[0,T]} \Big).$$

3. Show that the dual of the problem 2:

$$-\dot{\varphi} + a\varphi = 0, \quad 0 \leq t < t_N, \qquad \varphi(t_N) = e_N,$$

satisfies the stability estimate:

$$|\varphi(t)| \leq \begin{cases} \exp(\mathcal{A} t_N)|e_N|, & \text{if } |a(t)| \leq \mathcal{A} \quad \text{for all } 0 \leq t \leq t_N \\ |e_N|, & \text{if } a(t) \geq 0 \quad \text{for all } 0 \leq t \leq t_N. \end{cases}$$

4. Consider the boundary value problem

$$-(au')' = f, \quad 0 < x < 1, \qquad u(0) = u'(1) = 0.$$

(a) Show that the solution of this problem minimizes the energy integral

$$F(v) = \frac{1}{2}\int_0^1 a(v')^2 - \int_0^1 fv,$$

i.e., $u \in V$ where V is some function space and $F(u) = \min_{v \in V} F(v)$.

(b) Show that for $a = 1$ and a corresponding discrete minimum $F(U) = \min_{v \in V_h} F(v)$, with $U \in V_h \subset V$ we have that

$$F(U) = F(u) + \frac{1}{2}||(u - U)'||^2.$$

(c) Let $a = 1$ and show an a posteriori error estimate for the discrete energy minimum: i.e., for $|F(U) - F(u)|$, with V_h being the space of piecewise linear functions on subintervals of length h.

5. Apply the finite element method $cG(1)$ and prove an a posteriori error estimate for the following Neumann problem:

$$-\nabla \cdot (a \nabla u) + u = f, \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad a\partial_n u = g, \quad \text{on } \Gamma = \partial\Omega.$$

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