FEM08 - lecture 4

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Summary so far

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Error estimation

Equation: $R(u) = 0$.
Compute solution U, exact solution u.
Error: $e = u - U$
u is unknown, what can we say about the computational error e?

In science/engineering we want to guarantee our solution is close enough, give a tolerance on error:

$$\|e\| < TOL$$

We also want to compute U with as little computational power as possible.
We will see how a posteriori error estimates/bounds can make this possible.

Simple summary so far

Want to solve $R(u) = 0 \quad [-u'' - f = 0]$
Multiply with test function and integrate:
$\int_\Omega R(u) v dx = 0 \quad [\int_0^1 -u''v - fv dx = 0]$ (weak form)
Possibly use partial integration to move derivative to test function
$[\int_0^1 -u''v - fv dx = \int_0^1 u'v' - fv dx = 0, \quad v(0) = v(1) = 0]$
Linear case can define two parts, one with u and one without:
$\int_\Omega R(u) v dx = a(u, v) - L(v)$
$[\int_0^1 u'v' - fv dx \Rightarrow \quad a(u, v) = \int_0^1 u'v' dx, \quad L(v) = \int_0^1 fv dx]$
Pose finite element solution $U = \sum_{i=1}^M \xi_j \phi_j, \quad U \in V_h$
Note that we are seeking approximation of the solution function U and not directly the equation.
Plug in U, and approximation space V_h:
$a(U, v) - L(v) = 0, \quad \forall v \in V_h$

Simple summary so far

Plug in definitions of U and basis $\phi_j, \quad j = 1, ..., N$ of V_h to compute elements of matrix A and vector b

$$U = \sum_{i=1}^{M} \xi_j \phi_j, \quad U \in V_h$$

$$\Rightarrow$$

$$A_{ij} = a(\phi_j, \phi_i)dx,$$

$$b_i = L(\phi_j)$$

Solve linear system for ξ_i and construct solution U.

Error estimation

Examples for specific equations:
A priori example (order of convergence):

$$\|e\|_E \leq C\|hu''\|$$

A posteriori example (actual computable bound on error):

$$\|e\|_E \leq C\|hR(U)\|$$

Energy norm

For a linear PDE we observe:

$$a(u,v) - L(v) = 0$$
$$a(U,v) - L(v) = 0 \Rightarrow$$
$$a(u-U,v) = 0, \quad \forall v \in V_h$$

We can define the "energy" inner product/norm:

$$(f,g)_E = a(f,g) \Rightarrow \|f\|_E = \sqrt{a(f,f)}$$

For the equation:

$$R(u) = u'' - f = 0, \quad x \in (0,1) \Rightarrow$$
$$(f,g)_E = \int_0^1 f'g'dx \Rightarrow \|f\|_E = \int_0^1 (f')^2 dx$$

A priori estimation

(Recall estimate for L2 projection)

$$\begin{aligned}\|e\|_E^2 = (e,e)_E = (u-U,u-U)_E = \\ (u-U,u-U)_E + (u-U,v-v)_E = \\ (u-U,u-v)_E + (u-U,v-U)_E = \\ (u-U,u-v)_E \leq \|e\|_E \|u-v\|_E \Rightarrow \\ \|e\|_E \leq \|u-v\|_E, \quad \forall v \in V_h\end{aligned}$$

This proves that there is no better approximation than U in V_h in the energy norm.

A priori estimation

Continuing, remembering that interpolant $\pi u \in V_h$ and using interpolation estimate $\|u - \pi u\|_E \leq Ch\|u'\|_E$:

$$\|e\|_E \leq \|u - v\|_E, \quad \forall v \in V_h \Rightarrow$$
$$\|e\|_E \leq \|u - \pi u\|_E \leq Ch\|u'\|_E$$

Which means that the energy norm (in this case derivative) of the error converges to zero with first order rate.

A posteriori estimation

Want to extract $R(U)$ from expression with e.
Observe that $Ae = -R(U)$.
Galerkin orthogonality: $\int_0^1 U'v' - fv dx = 0, \quad \forall v \in V_h$
Note that $\pi e \in V_h$

$$\|e\|_E^2 = \int_0^1 e'e'dx = \int_0^1 (U'e' - fe)dx =$$
$$\int_0^1 (U'e' - fe - (U'\pi e' - f\pi e))dx =$$

A posteriori estimation

Continuing, using integration by parts on each cell/interval $K_i = [a_i, b_i], 0 < a_i < b_i < 1$ and that the interpolation error is zero in the nodes: $(e - \pi e)(x_j) = 0$.

$$\sum_{i=1}^{M} \int_{a_i}^{b_i} U'(e - \pi e)' dx - \int_0^1 f(e - \pi e) dx =$$

$$\sum_{i=1}^{M} \int_{a_i}^{b_i} (-U''(e - \pi e) - f(e - \pi e)) dx + [U'(e - \pi e)]_{a_i}^{b_i} =$$

Clean up, defining discontinuous $\hat{R}(U) = -U'' - f$

$$\|e\|_E^2 = \int_0^1 \hat{R}(U)(e - \pi e) dx$$

A posteriori estimation

Continuing using Cauchy-Schwartz and interpolation estimate $\|e - \pi e\| \leq Ch\|e'\| = Ch\|e\|_E$

$$\|e\|_E^2 = \int_0^1 \hat{R}(U)(e - \pi e)dx \leq \|\hat{R}(U)\|\|e - \pi e\| \leq$$
$$\|\hat{R}(U)\|Ch\|e\|_E$$

Which gives the final estimate/bound:

$$\|e\|_E \leq C\|h\hat{R}(U)\|$$

Note that the right hand side is computable given a solution U.

Adaptive algorithm

Want to compute the solution U to a given tolerance of the error:

$$\|e\|_E \leq TOL$$

Since we have:

$$\|e\|_E \leq C\|h\hat{R}(U)\|$$

We need to satisfy the requirement by the estimate:

$$C\|h\hat{R}(U)\| \leq TOL$$

Adaptive algorithm

Split up error bound into contributions from each cell:

$$C\|h\hat{R}(U)\| = C\sqrt{\int_0^1 (h\hat{R}(U))^2 dx} = C\sqrt{\sum_K \int_K (h\hat{R}(U))^2 dx}$$

Simple condition: refine (split) cells K where the error contribution $\int_K (h\hat{R}(U))^2 dx$ is largest.

Adaptive algorithm

Simple adaptive algorithm:

1. Choose an (arbitrary) initial triangulation T_h^0
2. Given the jth triangulation T_h^j compute FEM solution U
3. Compute residual $\hat{R}(U)$ and evaluate error bound $C\|h\hat{R}(U)\|$
4. If $C\|h\hat{R}(U)\| \leq TOL$ stop, else
5. Refine a percentage of cells K where the error contribution $\int_K (h\hat{R}(U))^2 dx$ is largest.