FEM08 - lecture 5

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Error control with FEniCS

- Error estimate
- Mesh refinement

Error estimation - duality

We've seen error estimates in energy norm: allows error control of a fixed global norm, i.e. $\|e\|_E < TOL$.

We are typically interested in a specific quantity:
$M(e) = (e, \psi) = \int_\Omega e\psi dx$
Example: $\psi = \chi_\omega / |\omega|, \quad \omega \subset \Omega, \quad \chi_\omega = 1, x \in \omega, \quad \chi_\omega = 0, x \notin \omega$
Gives average error in subset ω (close to an object for example)

We are looking for a posteriori estimate of the form:

$$|(e, \psi)| \leq C h^q \|\hat{R}(U)\| \ldots \tag{1}$$

Duality 1D

We have *primal equation*:

$-u'' - f = 0$

We introduce *dual equation*:

$-\phi'' - \psi = 0$

which is defined by:

$(Av, w) = (v, A^*w)$

where A is the diff. op. for the primal eq. and A^* for the dual eq. (in this case they are the same).

(Homogenous Dirichlet BC)

[Example from dual world]

Duality 1D

We compute solution U by Galerkin's method:
$\int_0^1 U'v' - fv dx = 0, \quad \forall v \in V_h, \quad U \in V_h$

Error $e = u - U$ satisfies:
$\int_0^1 e'w'dx = \int_0^1 -U'w' + fwdx$

Duality 1D

Want to bound quantity $M(e) = (e, \psi)$:

$$
\begin{aligned}
(e, \psi) = \int_0^1 e\psi dx = \int_0^1 e(-\phi'')dx = \\
\int_0^1 e'\phi' dx + [e\phi']_0^1 = \int_0^1 -U'\phi' + f\phi dx = \\
[GO] = \int_0^1 -U'(\phi - \pi\phi)' + f(\phi - \pi\phi)dx = \\
\sum_{K_i} \int_{a_i}^{b_i} (U'' + f)(\phi - \pi\phi)dx + [U'(\phi - \pi\phi)]_{a_i}^{b_i} = \\
\sum_{K_i} \int_{a_i}^{b_i} \hat{R}(U)(\phi - \pi\phi)dx
\end{aligned}
$$

Duality 1D

Use Cauchy-Schwartz as before:

$$
\begin{aligned}
|(e,\psi)| &\leq \|\hat{R}(U)\|\|\phi - \pi\phi\| \\
&\leq Ch^2\|\hat{R}(U)\|\|\phi''\|
\end{aligned}
$$

which is our estimate and where stability factor $S = \|\phi''\|$ gives information about how the residual (local error) grows.

Two options:
Try to estimate ϕ'' analytically or
Compute discretization of ϕ with FEM

In our case (Poisson) we have $\|\phi''\| = \|\psi\|$ where ψ is known data. In the general case we typically have to discretize ϕ.

A posteriori/duality 2D/3D

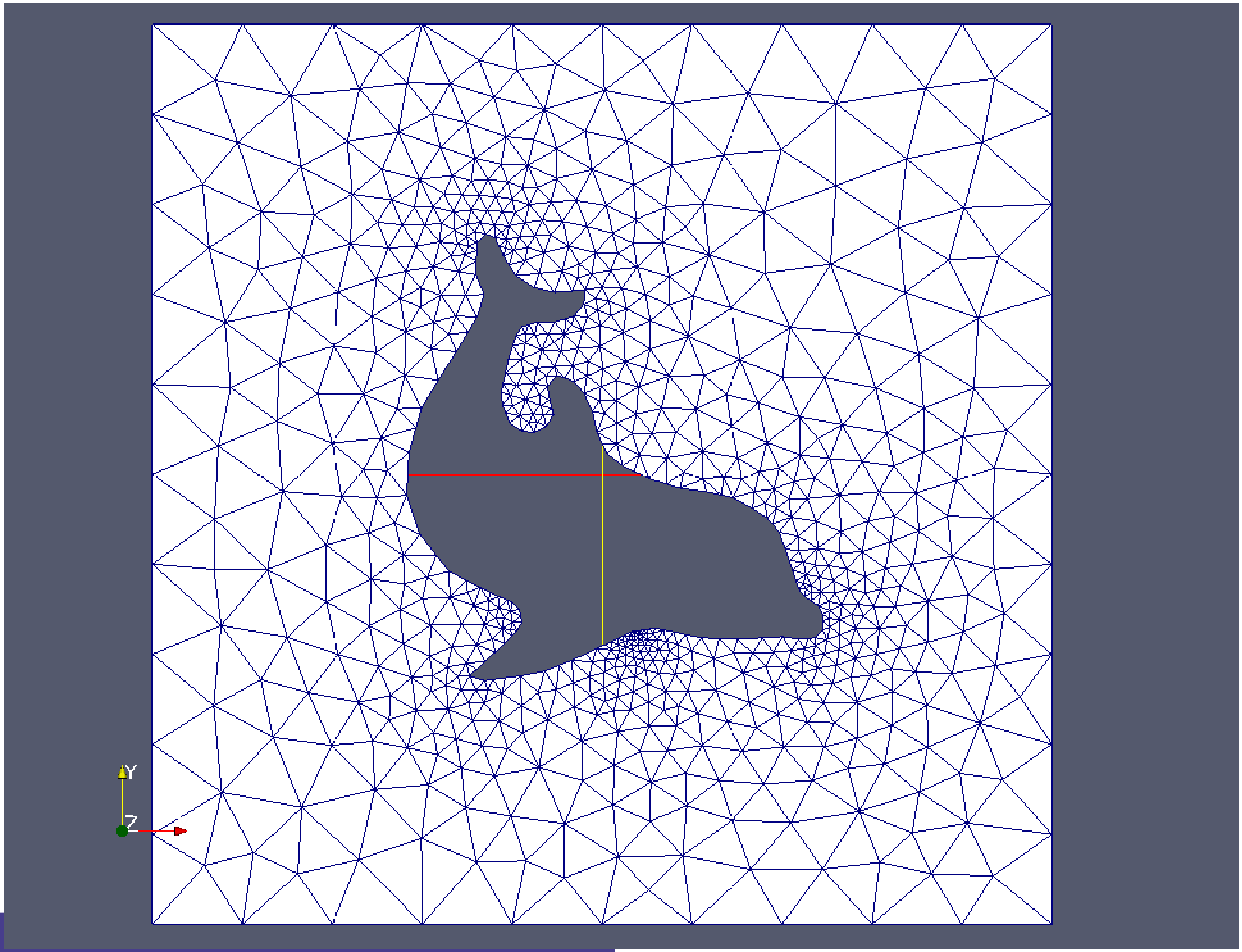
Mesh refinement

Recursive bisection (Rivara)

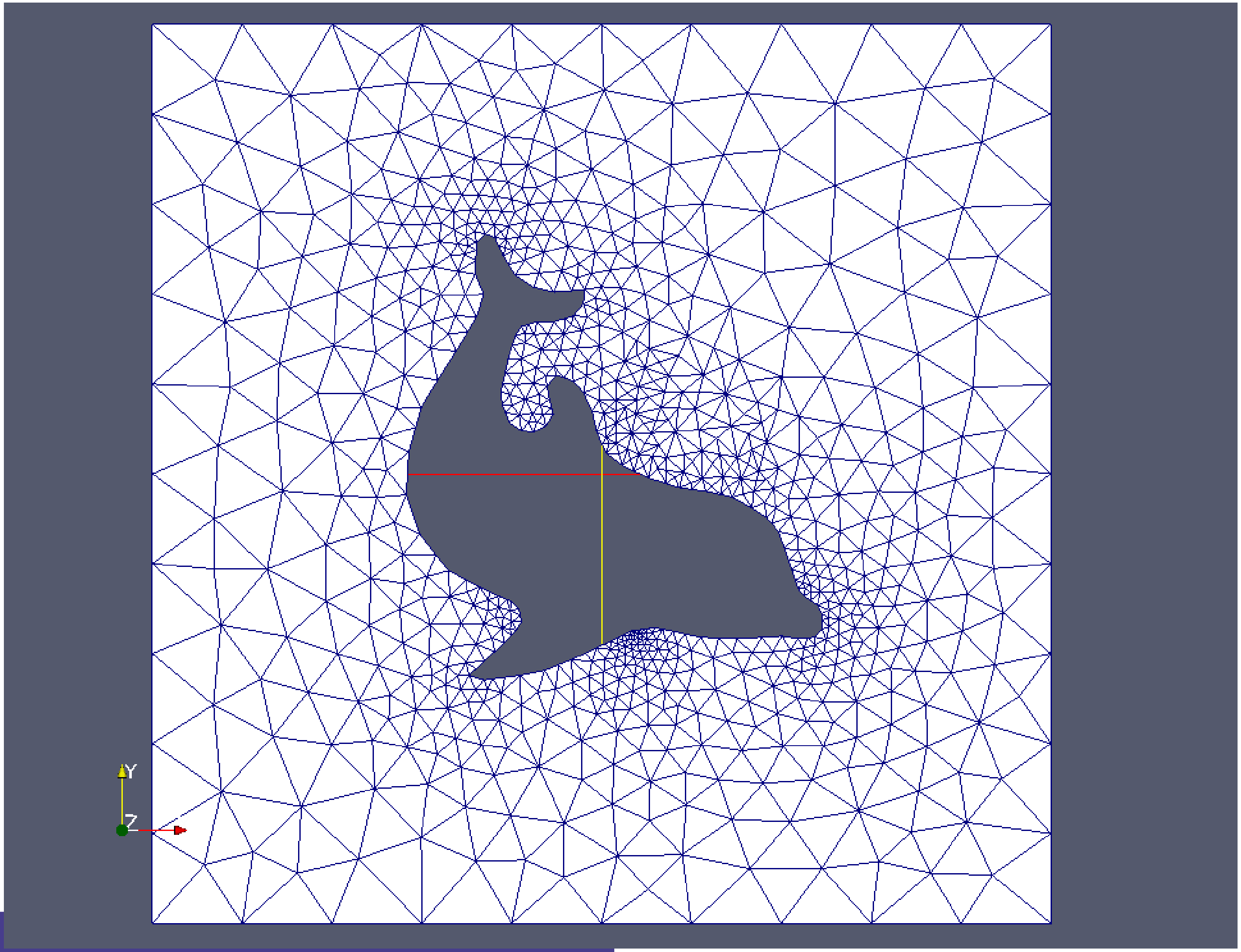
function bisect(K):
  Split longest edge e
  While $K_i(e)$ is non-conforming
    bisect(K_i)
where $K_i(e)$ is cell incident on edge e.

Same algorithm in 2D/3D. Bound on cell quality (smallest angle).

Mesh refinement



Mesh refinement



Mesh refinement