



# FEM08 - lecture 6

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# Contents

FEM for initial value problem (ODE)

Stability

FEM for initial boundary value problem (PDE)

# FEM for ODE

In general:

$$\dot{u} = f(t, u)$$

Model problem:

$$\dot{u} + au = f(t)$$

# FEM for ODE: dG(0)

dG(0):  $U \in W_k$  with  $U_0^- = u_0$ ,  $W_k$  space of piecewise constants

$$\int_0^T \dot{U}v + aUv + \sum_{n=1}^N (U_n - U_{n-1})v - fvd t = 0, \quad \forall v \in W_k \Rightarrow$$

$$U_n = U_{n-1} - \int_{t_{n-1}}^{t_n} a(t)U(t) + f(t)dt$$

$$U = \sum_{i=1}^N \xi_i \phi_i, \quad \phi_i = 1, \quad k_n = t_n - t_{n-1} \Rightarrow$$

$$\xi_n = \xi_{n-1} - \int_{t_{n-1}}^{t_n} a(t_n)\xi_n \phi_n + f(t_n)dt$$

# FEM for ODE: dG(0)

Right-point quadrature, we get backward Euler (+ quad. err.):

$$\xi_n = \xi_{n-1} - k_n(a(t_n)\xi_n + f(t_n)) + E_q$$

# FEM for ODE: cG(1)

cG(1):  $U \in V_k$  with  $U(0) = u_0$ ,  $V_k$  space of piecewise linears

$$\int_0^T \dot{U}v + aUv - fvd t = 0, \quad \forall v \in W_k \Rightarrow$$

$$U_n = U_{n-1} - \int_{t_{n-1}}^{t_n} a(t)U(t) + f(t)dt$$

$$U = \sum_{i=1}^N \xi_i \phi_i, \quad \phi_i = \frac{t - t_{i-1}}{t_i - t_{i-1}}, \quad k_n = t_n - t_{n-1} \Rightarrow$$

$$\xi_n = \xi_{n-1} - \int_{t_{n-1}}^{t_n} a(t_n)(\xi_{n-1}\phi_{n-1} + \xi_n\phi_n) + f(t_n)dt$$

# FEM for ODE: cG(1)

Trapezoid quadrature, we get Crank-Nicolson (+ quad. err.):

$$\xi_n = \xi_{n-1} - \frac{1}{2}k_n(a(t_{n-1})\xi_{n-1} + a(t_n)\xi_n + f(t_{n-1}) + f(t_n)) + E_q$$

# Error estimate for dG(0)

For general ODE:  $R(u) = \dot{u} - f(t, u) = 0$

Similar construction as for error estimates in space.

$$|u(T) - U(T)| \leq k |\hat{R}(U)| S(T)$$

where stability factor  $S(T) = \frac{\int_0^T |\dot{\phi}| dt}{e_T}$

$S$  gives a quantitative measure of the stability of the equation.



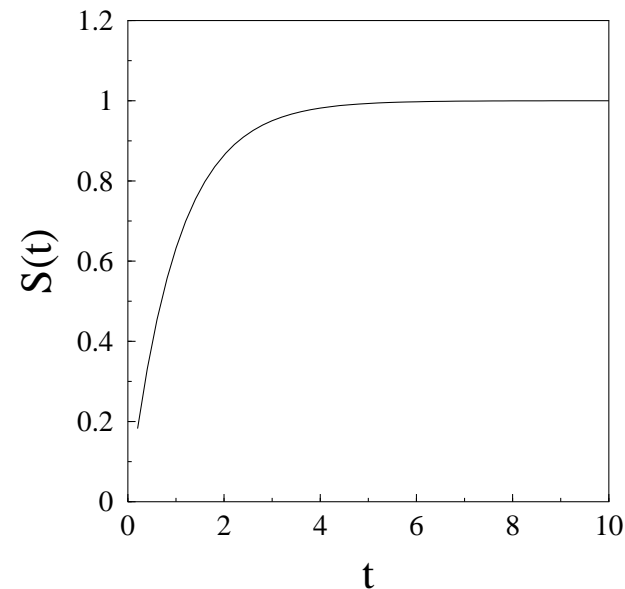
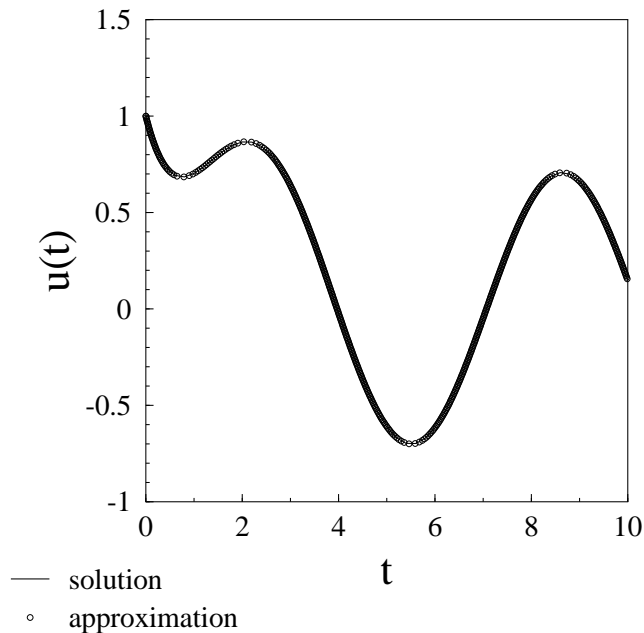
# Stability factor examples

Primal equation

$$\dot{u} + u = \sin(t), \quad u(0) = u_0$$

Dual equation

$$-\dot{\phi} + u = 0, \quad \phi(T) = e_T$$



$S$  doesn't grow with  $T \Rightarrow$  equation is very stable/*parabolic*.

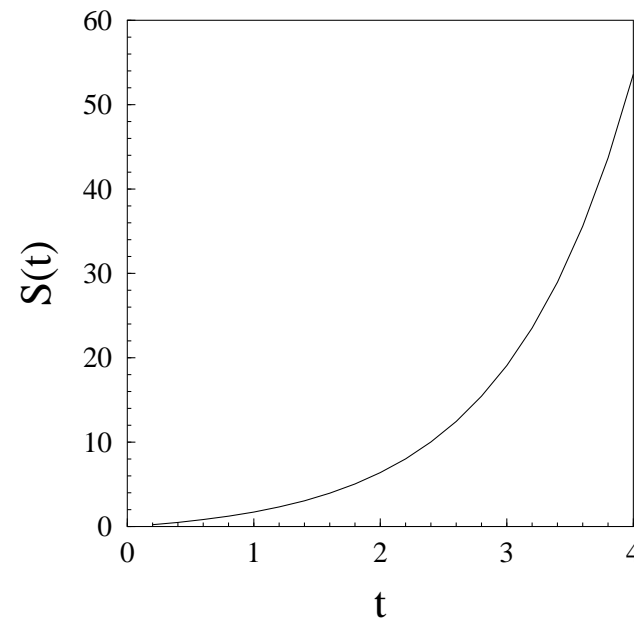
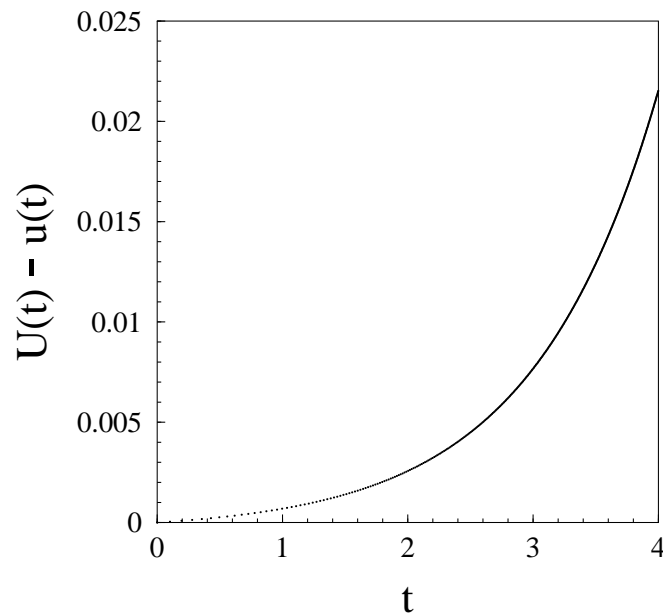
# Stability factor examples

Primal equation

$$\dot{u} - u = f(t), \quad u(0) = u_0$$

Dual equation

$$-\dot{\phi} - u = 0, \quad \phi(T) = e_T$$



$S$  grows exponentially with  $T \Rightarrow$  very unstable/expensive to compute accurately.

# FEM for IBVP/PDE (space-time)

Model problem (Heat equation):

$$u_t - \Delta u = f(t, x)$$

Domain  $D$  is cartesian product of domain in space and time interval:  $D = \Omega \times I$

Mesh is space-time slab  $D_n = T_n \times I_n$  where  $T_n = K$  is triangulation of  $\Omega$  and  $I_n$  is sub-interval of length  $k_n = t_n - t_{n-1}$ .

cG(1)dG(0): Trial and test space  $\bar{W}_k$  with basis functions  $\bar{\phi} = \phi_t \phi_x$ , where  $\phi_t$  basis functions of  $W_k$  (piecewise constant/discontinuous in time) and  $\phi_x$  basis functions of  $V_h$  (piecewise linear/continuous in space).

# FEM for IBVP/PDE (space-time)

Galerkin's method for cG(1)dG(0):

$$(U_n, v) = (U_{n-1}, v) - \int_{t_{n-1}}^{t_n} (\nabla U, \nabla v) + (f, v) dt, \quad \forall v \in \bar{W}_k$$

$$U = \sum_{i=1}^N \xi_i \bar{\phi} = \sum_{i=1}^N \xi_i \phi_{xi}, \quad v = \phi_{xi}, \quad i = 1, \dots, N$$

Again, with right-endpoint quadrature we get backward Euler (+quad. err.):

$$(U_n, v) = (U_{n-1}, v) - k_n (\nabla U(t_n, x), \nabla v) + (f(t_n), v), \quad \forall v \in \bar{W}_k$$

# FEM for IBVP/PDE (space-time)

Substituting  $U$  gives the formulas for matrix/vector elements.