# FEM08 - lecture 9

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# Contents

Remaining modules

Overview of course

Look ahead: Vector-valued / Non-linear PDE

Look ahead: Continuum mechanics / Navier-Stokes equations

# Adaptivity

- Want to compute solution $U$ so that $\|u - U\|_E \leq TOL$
- Use error estimate to determine *stopping criterion*: $\|Ch\hat{R}(U)\| \leq TOL$
- Utility function for computing residual provided
- Software instructions provided (cell-wise integral contribution, local refinement interface)

# Science - PDE

PDE: $A(u) = f, \quad x \in \Omega, \quad (x,t) \in \Omega \times I$
Residual: $R(U) = A(U) - f$

Examples:

- Poisson: $A(u) = -\Delta u$
- Convection-diffusion: $A(u) = \dot{u} + \beta \cdot \nabla u - \epsilon \Delta u$
- Heat: $A(u) = \dot{u} - \epsilon \Delta u$
- Wave: $A(u) = \ddot{u} - \epsilon \Delta u$

Weak/variational formulation:
Multiply by test function v and integrate
Find $u \in V$: $a(u,v) = L(v), \forall v \in V$
Existence and uniqueness by Lax-Milgram for class of PDE

# Galerkin's method

Choose finite element space $V_h \subset V$ (p-w linear polynomials on triangulation $T_h$) with basis $\phi_j{}_{j=1}^{M}$ (hat functions)

Find $U \in V_h$, $U = \sum_{i=1}^{M} \xi_i \phi_i$:
$a(U, v) = L(v), \forall v \in V_h$ (Galerkin orthogonality)
Galerkin optimal in energy norm $\|w\|_E = \sqrt{a(w, w)}$:
$\|u - U\|_E \leq \|u - v\|_E \forall v \in V_h$
$L_2$-projection:
Galerkin's method for $u = f$: $(U, v) = (f, v), \quad \forall v \in V_h$

# Assembly / discrete system

Plug in definitions of $U$ and basis $\phi_j, \quad j = 1, ..., M$ of $V_h$ to compute elements of matrix $A$ and vector $b$

$$A_{ij} = a(\phi_j, \phi_i)dx,$$
$$b_i = L(\phi_j)$$

Results in linear system $A\xi = b$

# Assembly / discrete system

Assembly algorithm:
for all cells $K \in \mathcal{T}$
    for all test functions $\hat{\varphi}_i$ on $K$
        for all trial functions $\varphi_j$ on $K$
            1. Compute $I = a(\varphi_j, \hat{\varphi}_i)_K$
            2. Add $I$ to $(A_h)_{ij}$
        end
    end
end

Use mapping $F(X) = x^1\varphi_1^0(X) + x^2\varphi_2^0(X) + x^3\varphi_3^0(X)$ to simplify: only need to construct basis functions and integrate on the reference element $K_0$

# Error estimation

A priori example (order of convergence):

$$\|e\|_E \leq \|u - v\|_E = \|u - \pi u\| \leq C\|hu''\|$$

A posteriori example (actual computable bound on error):

$$\|e\|_E \leq C\|hR(U)\|$$

A posteriori example with duality (measure quantity of error):

$$\begin{aligned} |(e, \psi)| &\leq \|\hat{R}(U)\|\|\phi - \pi\phi\| \\ &\leq Ch^2\|\hat{R}(U)\|\|\phi''\| \end{aligned}$$

Dual equation $A^*(\phi) = \psi$ with $(Av, w) = (v, A^*w), \forall v, w \in V$

Interpolation estimates of form: $\|u - \pi u\| \leq Ch^2\|u''\|$, remember $\pi u \in V_h$

# Adaptivity

Want to compute the solution $U$ to a given tolerance of the error:

$$\|e\|_E \leq TOL$$

1. Choose an (arbitrary) initial triangulation $T_h^0$
2. Given the $j$th triangulation $T_h^j$ compute FEM solution $U$
3. Compute residual $\hat{R}(U)$ and evaluate error bound $C\|h\hat{R}(U)\|$
4. If $C\|h\hat{R}(U)\| \leq TOL$ stop, else
5. Refine a percentage of cells $K$ where the error contribution $\int_K (h\hat{R}(U))^2 dx$ is largest.

# Written exam

Examination split up into two parts with half of grade each:

- Module submissions
- Written examination (closed-book)

Scope is the same for both parts (look at module contents and lecture notes).
Purpose of written examination is to test understanding of concepts in course, not memorization.

# Look ahead: Vector-valued PDE

We have looked at scalar-valued PDE where $u$ and $v$ are scalar-valued functions.
Consider $u$ a vector-valued function $u = [u_1, u_2]$ (velocity for example) and $v = [v_1, v_2]$.
Define $L_2$ inner product: $(u, v) = \int_\Omega u \cdot v dx$
We can define PDE: $A(u) = f$
Weak form: $a(u, v) = L(v)$
Lagrange basis functions:

$$\begin{aligned}
\hat{\phi}_1 &= (\phi_1, 0) \\
\hat{\phi}_2 &= (0, \phi_1) \\
&\dots \\
\hat{\phi}_{2N-1} &= (\phi_N, 0) \\
\hat{\phi}_{2N} &= (0, \phi_N)
\end{aligned}$$

# Non-linear PDE

If the problem is nonlinear, for example,

$$-\nabla \cdot (|\nabla u|\ \nabla u) = f,$$

we rewrite the problem in fixed-point form with $\tilde{u} = u^{n-1}$, $u = u^n$

$$-\nabla \cdot (|\nabla \tilde{u}|\ \nabla u) = f.$$

As before, we obtain a linear system $A_h \xi^n = b$, but now

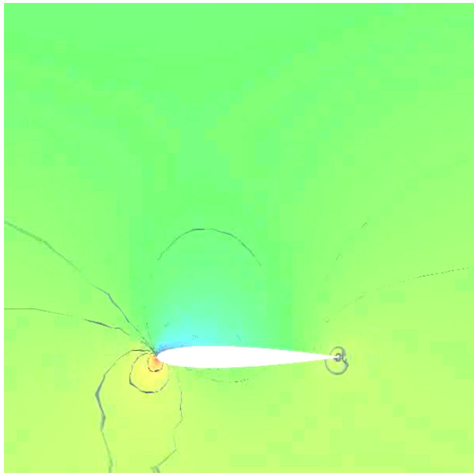
$$A_h = A_h(\tilde{u}) = A_h(\xi^{n-1}),$$

i.e. $A_h(\xi^{n-1})\xi^n = f$.

Newton's method is a systematic way of choosing the optimal fixed-point formulation.

# Continuum mechanics / NSE

## Incompressible Navier-Stokes



$$
\begin{aligned}
R_1(u) &= \dot{u} + u \cdot \nabla u - \nu \Delta u + \nabla p - f = 0 \\
R_2(u) &= \nabla \cdot u = 0
\end{aligned}
$$

$\hat{u} = [u, p]$

$u : \Omega \times I \rightarrow R^3$ velocity $p : \Omega \times I \rightarrow R$ pressure

Test function:

$\hat{v} = [v, q]$

Galerkin:

$(R(\hat{u}), \hat{v}) = 0, \forall \hat{v} \in \hat{V}_h$

Stabilization:

$(R(\hat{u}), \hat{v} + \delta R(\hat{v})) = 0, \forall \hat{v} \in \hat{V}_h$

# ALE

$$
\begin{aligned}
x &= \phi(t,\bar{x}) \\
D_t\phi(t,\bar{x}) &= \hat{v}(t,\bar{x}) \\
u(t,x) &= u(t,\phi(t,\bar{x})) \\
&\Rightarrow \\
D_{(t,x)}u(t,x) &= D_{(t,\phi)}u(t,\phi(t,\bar{x}))D_{(t,\bar{x})}\phi(t,\bar{x}) \Rightarrow \\
D_t u(\bar{x}) &= D_t u(x) + D_\phi u \cdot D_t\phi \\
&= D_t u + D_\phi u \cdot \hat{v} = D_t u + \hat{v} \cdot \nabla_\phi u
\end{aligned}
\tag{1}
$$

$$
\dot{u}(x) + \beta \cdot \nabla u = \dot{u}(\bar{x}) + (\beta - \hat{v}) \cdot \nabla u \tag{2}
$$

# Project course

Project course in scientific computing period 2 2008 (DN2295). Choose (or come up with) project plan and carry out project with supervision from members of our research group (me, Johan Hoffman, ...).