DN2660 The Finite Element Method: Written Examination
Tuesday 2008-10-21, 14-19
Coordinator: Johan Jansson

Aids: none   Time: 5 hours

Answers must be given in English. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be given zero points, while a good explanation with a wrong answer can give some points. Maximum is 30 points.

Good luck,
Johan

Problem 1 - Galerkin's method

Consider the equation:

$$-\nabla \cdot (a(x)\nabla u(x)) + b(x)u(x) = f(x), \quad x \in \Omega$$
$$u(x) = 0, \quad x \in \Gamma$$

where $a(x), b(x)$ and $f(x)$ are known coefficients.

Recall the formula:

$$\int_\Omega D_{x_i} vw dx = \int_\Gamma vwn_i ds - \int_\Omega vD_{x_i} wdx, \quad i = 1, 2, ..., d$$

1. (2p) Formulate a finite element method (Galerkin's method) for the equation using piecewise linear approximation (cG(1)).
2. (1p) Explain what the Galerkin orthogonality means, both in general and for this equation.
3. (1p) Change the boundary condition to homogenous Neumann, and show the effect on the FEM formulation.
4. (1p) Discuss using piecewise polynomial basis functions versus using global basis functions with regard to the resulting linear system and adaptivity.

Solution 1

1. (2p)

   Multiply by test function $v \in V^0$ with V^0 defined as $v(x) = 0, x \in \Gamma$ due to the Dirichlet boundary condition, and use integration by parts: $(-\nabla\cdot(a(x)\nabla u(x)), v(x)) = (a(x)\nabla u(x), \nabla v(x)) - \int_\Gamma(\nabla u(x) \cdot n)v(x)ds$ where the boundary term is zero due to the condition on v

   $$(R(u), v) = (a(x)\nabla u(x), \nabla v(x)) + (b(x)u(x), v(x)) - (f(x), v(x)) = 0, \quad x \in \Omega, \forall v \in V^0$$
   $$u(x) = 0, \quad x \in \Gamma$$

   (1p)

   Seek approximation $U = \sum_{i=1}^{N} \xi_i \phi_i \in V_h^0$ with $(R(U), v) = 0, \quad \forall v \in V_h^0$.

   Thus:

   $$(R(U), v) = (a(x)\nabla U(x), \nabla v(x)) + (b(x)U(x), v(x)) - (f(x), v(x)) = 0, \quad x \in \Omega, \quad \forall v \in V_h$$
   $$U(x) = 0, \quad x \in \Gamma$$

   (1p)

2. (1p)

   The Galerkin Orthogonality is the condition $(R(U), v) = 0, \forall v \in V_h$ we enforce on U. See above for the formulation for this equation.

3. (1p)

   Homogenous Neumann means $-\nabla u(x) \cdot n = 0$. The formulation is the same as for homogenous Dirichlet above except we now don't have a condition on v on the boundary, so V and V_h are defined differently. The boundary term is still zero due to the boundary condition definition. Thus:

   $$(R(U), v) = (a(x)\nabla U(x), \nabla v(x)) + (b(x)U(x), v(x)) - (f(x), v(x)) = 0, \quad x \in \Omega, \quad \forall v \in V_h$$
   $$-\nabla U(x) \cdot n = 0, \quad x \in \Gamma$$

4. (1p)

   Piecewise polynomial basis functions have local support (only integrals on cells incident to the node defining the basis function are non-zero), which means that the matrix will be sparse (many elements are zero). Global basis functions have global support, which means that the matrix will be dense (possibly all elements are non-zero). Thus global basis functions will be more expensive per degree of freedom. (0.5p)

   Piecewise polynomial basis functions allow local refinement, where we can subdivide a cell to locally get a more accurate representation. With global basis functions we can only increase the order, which has effect on the whole domain. Thus local refinement is not possible. (0.5p)

Problem 2 - Stability

Consider the heat equation with zero source:

$$\begin{aligned} \dot{u} - \Delta u = 0, &\quad x \in \Omega, \quad t \in [0, T] \\ u(0, x) = u_0(x) & \\ u(t, x) = 0, &\quad x \in \Gamma \end{aligned}$$

1. (2p) Derive the stability estimate (hint: multiply by u):

$$\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2 dt = \|u(0)\|^2$$

   Explain what a stability estimate is in general, and give an interpretation what this particular stability estimate says about the temperature u.

2. (2p) Explain the basic concept behind a streamline diffusion stabilized finite element method.

Solution 2

1. (2p)

   See module 8 for derivation of the stability estimate.

   Generally a stability estimate bounds the solution or derivatives of the solution (u, ∇u) in terms of data (f, u_0). If we have a stability estimate we can be sure that the solution does not grow uncontrollably and we can use this property in further error estimation.

   In this specific case we can see that since all terms are positive, the norm of the temperature $\|u(t)\|^2$ can never increase in time.

2. (2p)

   See module 8 for an explanation of the concept behind streamline diffusion.

Problem 3 - Assembly of a linear system

1. (3p) Formulate a general assembly algorithm of a linear system given a bilinear form $a(u, v)$ and linear form $L(v)$ representing a linear boundary value partial differential equation (PDE) in 2D/3D, with a piecewise linear Galerkin approximation (cG(1)). Include explanations of the following concepts:

   - Mesh
   - Map from reference cell

- Formula for computation of a matrix and vector element
- Quadrature

2. (2p) Define a basic linear boundary value PDE in 1D. Apply Galerkin's method, construct a simple mesh and compute a matrix element by hand (you don't have to use a general assembly algorithm here).

Solution 3

1. (3p)

   See module 4 for a description of a general assembly algorithm.

2. (2p)

   See CDE chapter 8 for an example of assembly of a boundary value PDE in 1D.

Problem 4 - Error estimation

Consider the equation:

$$-u'' + u = f, \quad x \in [0, 1]$$
$$u(0) = u(1) = 0$$

We define the energy norm for this equation: $\|w\|_E = \sqrt{a(w, w)} = \sqrt{\int_0^1 (w')^2 + w^2 dx}$

1. (2p) Show that Galerkin's method is optimal for the equation and derive an a priori error estimate in the energy norm $\|w\|_E$.
2. (2p) Derive an a posteriori error estimate in the energy norm (hint: you can use that $\|w\|_{L_2} \leq \|w\|_E$).
3. (2p) Sketch the basic steps for how to construct an error estimate of a general quantity of the error (e, ψ) using duality.

Solution 4

1. (2p)

$$\begin{aligned}
\|e\|_E^2 = (e, e)_E = (u - U, u - U)_E = \\
(u - U, u - U)_E + (u - U, v - v)_E = \\
(u - U, u - v)_E + (u - U, v - U)_E = \\
(u - U, u - v)_E \leq \|e\|_E \|u - v\|_E \Rightarrow \\
\|e\|_E \leq \|u - v\|_E, \quad \forall v \in V_h
\end{aligned}$$

This proves that there is no better approximation than U in V_h in the energy norm (if we can define the energy norm).

Continuing, remembering that interpolant $\pi u \in V_h$ and using interpolation estimate $\|u - \pi u\|_E \le Ch\|u'\|_E$:

$$\|e\|_E \le \|u - v\|_E, \quad \forall v \in V_h \Rightarrow$$
$$\|e\|_E \le \|u - \pi u\|_E \le Ch\|u'\|_E$$

Which means that the energy norm (in this case derivative) of the error converges to zero with first order rate.

2. (2p)

We want to extract $R(U)$ from expression with $e = u - U$.

Observe that $\int_0^1 e'w' + ewdx = \int_0^1 -U'w' - Uw + fwdx$.

Galerkin orthogonality: $\int_0^1 U'v' + Uv - fvdx = 0, \quad \forall v \in V_h$ Note that $\pi e \in V_h$

$$\|e\|_E^2 = \int_0^1 e'e'dx = \int_0^1 (-U'e' - Ue + fe)dx =$$
$$\int_0^1 (-U'e' - Ue + fe - (-U'\pi e' - U\pi e + f\pi e))dx =$$

Continuing, using integration by parts on each cell/interval $K_i = [a_i, b_i], 0 < a_i < b_i < 1$ and that the interpolation error is zero in the nodes: $(e - \pi e)(x_j) = 0$.

$$\sum_{i=1}^{M} \int_{a_i}^{b_i} -U'(e - \pi e)'dx - U(e - \pi e) + f(e - \pi e)dx =$$
$$\sum_{i=1}^{M} \int_{a_i}^{b_i} (U''(e - \pi e) - U(e - \pi e) + f(e - \pi e))dx + [U'(e - \pi e)]_{a_i}^{b_i} =$$

Clean up, defining discontinuous $\hat{R}(U) = U'' - U + f$

$$\|e\|_E^2 = \int_0^1 \hat{R}(U)(e - \pi e)dx$$

Continuing using Cauchy-Schwartz and interpolation estimate $\|e-\pi e\| \leq Ch\|e'\| \leq Ch\|e\|_E$

$$\|e\|_E^2 = \int_0^1 \hat{R}(U)(e-\pi e)dx \leq \|\hat{R}(U)\|\|e-\pi e\| \leq \|\hat{R}(U)\|Ch\|e\|_E$$

Which gives the final estimate/bound:

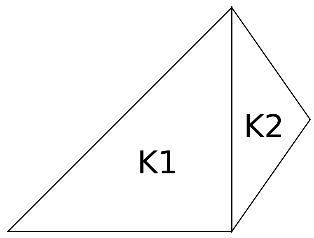
$$\|e\|_E \leq C\|h\hat{R}(U)\|$$

3. (2p) See module 4.

Problem 5 - Adaptivity

1. (3p) Formulate an adaptive finite element method based on an a posteriori error estimate with local mesh refinement given a tolerance TOL on a quantity or norm of the error $e = u - U$. Discuss why adaptivity is important.

2. (2p) Formulate the Rivara recursive bisection algorithm. Consider the mesh:



Mark the triangle K2 for refinement and perform the Rivara algorithm by hand, show all steps.

Solution 5

1. (3p)

See module 6 for a formulation of an adaptive algirithm. (2p)

Adaptivity is important because it can greatly improve efficiency. If we don't have adaptivity we must refine the mesh uniformly (everywhere) to be sure that the error converges. If the error contribution is localized, this efficiency difference could be enormous. (1p)

2. (2p)

   See module 6 for a formulation of the Rivara algorithm. (1p)

   First we call bisect(K2), where we will bisect the longest edge of K2, the edge e_{12} between K1 and K2, creating two new cells K3 and K4. We check if all cells incident to the edge are conforming, and see that K1 is not conforming because there is a hanging node on the edge e_{12}.

   We thus call bisect(K1), where we will bisect the longest edge (diagonal edge) of K1, thus creating two new cells, K5 and K6 where K6 is incident to e_{12}. We see that K6 is not conforming because there is still a hanging node on the edge e_{12}.

   We thus call bisect(K6), where we bisect the edge e_{12}, thus creating two new cells K7 and K8, thus eliminating the hanging node on e_{12}. We now have no further hanging nodes and the original bisect(K2) call will return. (1p)

Problem 6 - Abstract formulation

1. (2p) Explain what the Lax-Milgram theorem says, what it requires to be satisfied, and what it can be used for.

2. (2p) Assume that we have, in an abstract formulation, a boundary value PDE: $a(u, v) = L(v), \quad \forall v \in V$

   We apply Galerkin's method and construct a discrete solution U which satisfies: $a(U, v) = L(v), \quad \forall v \in V_h$

   Show that the discrete solution U is optimal in the energy norm: $\|w\|_E = \sqrt{a(w, w)}$.

   We assume we can find a solution and that the energy norm exists.

Solution 6

1. (2p)

   See module 9 for a formulation and explanation of the Lax-Milgram theorem. The theorem can be used to prove existence and uniqueness of solutions to linear, elliptic boundary value PDE.

2. (2p)

   See CDE chapter 21.