FEM09

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Course overview

- Science - differential equations
- Function approximation using polynomials
- Galerkin’s method (finite element method)
- Assembly of discrete systems
- Error estimation
- Mesh operations
- Stability
- Existence and uniqueness of solutions

Course structure

- Course divided into self-contained modules (from preceding slide)
- Module:
  - Theory
  - Software
  - Write report (theory + software)

Science - modeling

Science: modeling (formulating equations) + computation (solving equations)

- Model natural laws (primarily) in terms of differential equations
- Partial differential equation (PDE):

$$A(u(x)) = f, \quad x \in \Omega$$

with A differential operator.

Initial value problem $u(x_0) = g$ (x is “time”, $\Omega = [0, T]$)

Boundary value problem $u(x) = g, \quad x \in \Gamma$ or
$(\nabla u(x)) \cdot n = g, \quad x \in \Gamma$ (x is “space”)

Boundary value problem $u(x) = g, \quad x \in \Gamma$ (x is “space”)

Initial boundary value problem Both are also possible

Science - computation

Finite Element Method (FEM): approximate solution function u as (piecewise) polynomial.

Compute coefficients by enforcing orthogonality (Galerkin's method).

Implement general algorithms for arbitrary differential equations

In this course we will use and understand a general implementation for discretizing PDE with FEM: FEniCS using the Python programming language.

Free software / Open source implementations

Science/FEM - examples

Newton's 2nd law: $F = ma, u = (u_1, u_2)$:

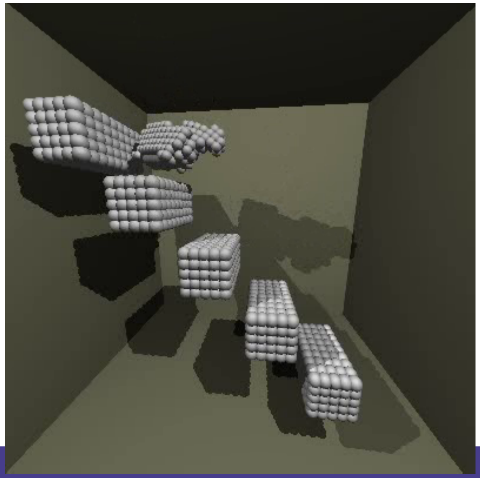
$$\begin{aligned} \dot{u_1}(t) &= u_2(t) \\ \dot{u_2}(t) &= F(u(t)) \\ u(0) &= u_0, \quad t \in [0, T] \end{aligned}$$

Science/FEM - examples

Incompressible Navier-Stokes

$$\begin{aligned} \dot{u} + u \cdot \nabla u - \nu \Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned}$$

Elasticity - solid mechanics



$$\dot{u} + \nabla \cdot \sigma = f$$

Polynomial approximation

Systematic method for computing approximate solutions:

We seek polynomial approximations U to u.

A vector space can be constructed with set of polynomials on domain (a, b) as basis vectors, where function addition and scalar multiplication satisfy the requirements for a vector space.

We can also define an inner product space with the L_2 inner product defined as:

$$(f, g)_{L_2} = \int_\Omega f(x)g(x)dx$$

Polynomial approximation

The L_2 inner product generates the L_2 norm:

$$\|f\|_{L_2} = \sqrt{(f,f)_{L_2}}$$

Just like in R^d we define orthogonality between two vectors as:

$$(f,g)_{L_2} = 0$$

We also have Cauchy-Schwartz inequality:

$$|(f,g)_{L_2}| \leq \|f\|_{L_2}\|g\|_{L_2}$$

Basis

We call our polynomial vector space $V^q = P^q(a,b)$ consisting of polynomials:

$$p(x) = \sum_{i=0}^{q} c_i x^i$$

One basis is the monomials: $\{1, x, ..., x^q\}$

Equation

What do we mean by equation?

We define the *residual* function $R(U)$ as:
$R(U) = A(U) - f$

We can thus define an *equation* with exact solution u as:
$R(u) = 0$

Galerkin's method

We seek a solution U in finite element vector space V^q of the form:

$$U(x) = \sum_{j=1}^{M} \xi_j \phi_j(x)$$

We require the residual to be orthogonal to V^q:

$$(R(U), v) = 0, \forall v \in V^q$$

Appendix

Example - heat equation

Thin wire occupying $x \in [0, 1]$ heated by a heat source $f(x)$.
We seek stationary temperature $u(x)$.
Let $q(x)$ be heat flux along positive x-axis.
Conservation of energy in arbitrary sub-interval:

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x)\, dx.$$

Fundamental theorem of calculus:

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} q'(x)\, dx,$$

Together:

$$\int_{x_1}^{x_2} q'(x)\, dx = \int_{x_1}^{x_2} f(x)\, dx.$$

Since the sub-interval is arbitrary:

$$q'(x) = f(x) \quad \text{for } 0 < x < 1,$$

Example - heat equation

Constitutive law - Fourier's law:

$$q(x) = -a(x)u'(x),$$

Inserting gives the heat equation:

$$-(a(x)u'(x))' = f(x) \quad \text{for } 0 < x < 1.$$

Lagrange (nodal) Basis

We will use the Lagrange basis: $\{\lambda_i\}_{i=0}^{q}\}$ associated to the distinct points $\xi_0 < \xi_1 < ... < \xi_q$ in (a, b), determined by the requirement that $\lambda_i(\xi_j) = 1$ if $i = j$ and 0 otherwise.

$$\lambda_i(x) = \prod_{j \neq i} \frac{x - \xi_j}{\xi_i - \xi_j}$$

$$\lambda_0(x) = (x - \xi_1)(\xi_0 - \xi_1)$$

$$\lambda_1(x) = (x - \xi_0)/(\xi_1 - \xi_0)$$

Polynomial interpolation

We assume that f is continuous on $[a, b]$ and choose distinct interpolation nodes $a \leq \xi_0 < \xi_1 < \cdots < \xi_q \leq b$ and define a polynomial interpolant $\pi_q f \in \mathcal{P}^q(a, b)$, that interpolates $f(x)$ at the nodes $\{\xi_i\}$ by requiring that $\pi_q f$ take the same values as f at the nodes, i.e. $\pi_q f(\xi_i) = f(\xi_i)$ for $i = 0, ..., q$. Using the Lagrange basis corresponding to the ξ_i, we can express $\pi_q f$ using "Lagrange's formula":

$$\pi_q f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) + \cdots + f(\xi_q)\lambda_q(x) \quad \text{for } a \leq x \leq b$$

Interpolation error

Mean value theorem:

$$f(x) = f(\xi_0) + f'(\eta)(x - \xi_0) = \pi_0 f(x) + f'(\eta)(x - \xi_0)$$

for some η between ξ_0 and x, so that

$$|f(x) - \pi_0 f(x)| \le |x - \xi_0| \max_{[a,b]} |f'| \quad \text{for all } a \le x \le b$$

Giving:

$$\|f - \pi_0 f\|_{L_2(a,b)} \le C_i (b - a) \|f'\|_{L_2(a,b)}$$

L_2 projection

We seek a polynomial approximate solution $U \in P^q(a,b)$ to the equation:

$$R(u) = u - f = 0, \quad x \in (a,b)$$

where f in general is not polynomial, i.e. $f \notin P^q(a,b)$.
This means $R(U)$ can in general not be zero. The best we can hope for is that $R(U)$ is orthogonal to $P^q(a,b)$ which means solving the equation:

$$(R(U), v)_{L_2} = (U - f, v)_{L_2} = 0, \quad x \in \Omega, \quad \forall v \in P^q(a,b)$$

Error estimate

The orthogonality condition means the computed L_2 projection U is the best possible approximation of f in $P^q(a,b)$ in the L_2 norm:

$$\|f-U\|^2 = (f-U, f-U) =$$

$$(f-U, f-v) + (f-U, v-U) =$$

$$[v-U \in P^q(a,b)] = (f-U, f-v) \leq \|f-U\|\|f-v\|$$

$$\Rightarrow$$

$$\|f-U\| \leq \|f-v\|, \quad \forall v \in P^q(a,b)$$

Error estimate

Since $\pi f \in P^q(a,b)$, we can choose $v = \pi f$ which gives:

$$\|f - U\| \leq \|f - \pi f\|$$

i.e. we can use an interpolation error estimate since it bounds the projection error.