FEM08 - lecture 5

Johan Jansson

jjan@csc.kth.se

CSC

KTH

Contents

Overview of course so far

Error estimation:

- Duality and a posteriori
- Adaptivity

Mesh algorithms

- Mesh refinement

Error control with FEniCS

- Error estimate
- Mesh refinement

Error estimation - duality

We've seen a posteriori error estimates in energy norm: allows error control of a fixed global norm, i.e. $\|e\|_E < TOL$.

We are typically interested in a specific quantity of interest: $M(e) = (e, \psi) = \int_\Omega e\psi dx$

Example: $\psi = \chi_\omega / |\omega|, \quad \omega \subset \Omega, \quad \chi_\omega = 1, x \in \omega, \quad \chi_\omega = 0, x \notin \omega$
Gives average error in subset ω (close to an object for example)

We are looking for a posteriori estimate of the form:

$$|(e, \psi)| \leq Ch^q \|\hat{R}(U)\| ... \quad (1)$$

Duality 1D

We have *primal equation*:
$-u'' - f = 0$
We introduce *dual equation*:
$-\phi'' - \psi = 0$
which is defined by:
$(Av, w) = (v, A^* w)$
where A is the diff. op. for the primal eq. and A^* for the dual eq.
(in this case they are the same).
(Homogenous Dirichlet BC)

[Example from dual world]

Duality 1D

We compute solution U by Galerkin's method:
$\int_0^1 U'v' - fvdx = 0, \quad \forall v \in V_h, \quad U \in V_h$

Error $e = u - U$ satisfies:
$\int_0^1 e'w'dx = \int_0^1 -U'w' + fwdx$

Duality 1D

Want to bound quantity $M(e) = (e, \psi)$:

$$
\begin{aligned}
(e,\psi) &= \int_0^1 e\psi dx = \int_0^1 e(-\phi'')dx = \\
&\int_0^1 e'\phi' dx + [e\phi']_0^1 = \int_0^1 -U'\phi' + f\phi dx = \\
&[GO] = \int_0^1 -U'(\phi - \pi\phi)' + f(\phi - \pi\phi)dx = \\
&\sum_{K_i} \int_{a_i}^{b_i} (U'' + f)(\phi - \pi\phi)dx + [U'(\phi - \pi\phi)]_{a_i}^{b_i} = \\
&\sum_{K_i} \int_{a_i}^{b_i} \hat{R}(U)(\phi - \pi\phi)dx
\end{aligned}
$$

Duality 1D

Use Cauchy-Schwartz as before:

$$
\begin{aligned}
|(e,\psi)| &\leq \|\hat{R}(U)\| \|\phi - \pi\phi\| \\
&\leq Ch^2 \|\hat{R}(U)\| \|\phi''\|
\end{aligned}
$$

which is our estimate and where stability factor $S = \|\phi''\|$ gives information about how the residual (local error) grows.

Two options:
Try to estimate ϕ'' analytically or
Compute discretization of ϕ with FEM

In our case (Poisson) we have $\|\phi''\| = \|\psi\|$ where ψ is known data. In the general case we typically have to discretize ϕ.

Adaptive algorithm

Want to compute the solution U to a given tolerance of the error (energy norm in this case):

$$\|e\|_E \leq TOL$$

Since we have:

$$\|e\|_E \leq C\|h\hat{R}(U)\|$$

We need to satisfy the requirement by the estimate:

$$C\|h\hat{R}(U)\| \leq TOL$$

Adaptive algorithm

Split up error bound into contributions from each cell:

$$C\|h\hat{R}(U)\| = C\sqrt{\int_0^1 (h\hat{R}(U))^2 dx} = C\sqrt{\sum_K \int_K (h\hat{R}(U))^2 dx}$$

Simple condition: refine (split) cells K where the error contribution $\int_K (h\hat{R}(U))^2 dx$ is largest.

Adaptive algorithm

Simple adaptive algorithm:

1. Choose an (arbitrary) initial triangulation T_h^0
2. Given the jth triangulation T_h^j compute FEM solution U
3. Compute residual $\hat{R}(U)$ and evaluate error bound $C\|h\hat{R}(U)\|$
4. If $C\|h\hat{R}(U)\| \leq TOL$ stop, else
5. Refine a percentage of cells K where the error contribution $\int_K (h\hat{R}(U))^2 dx$ is largest.

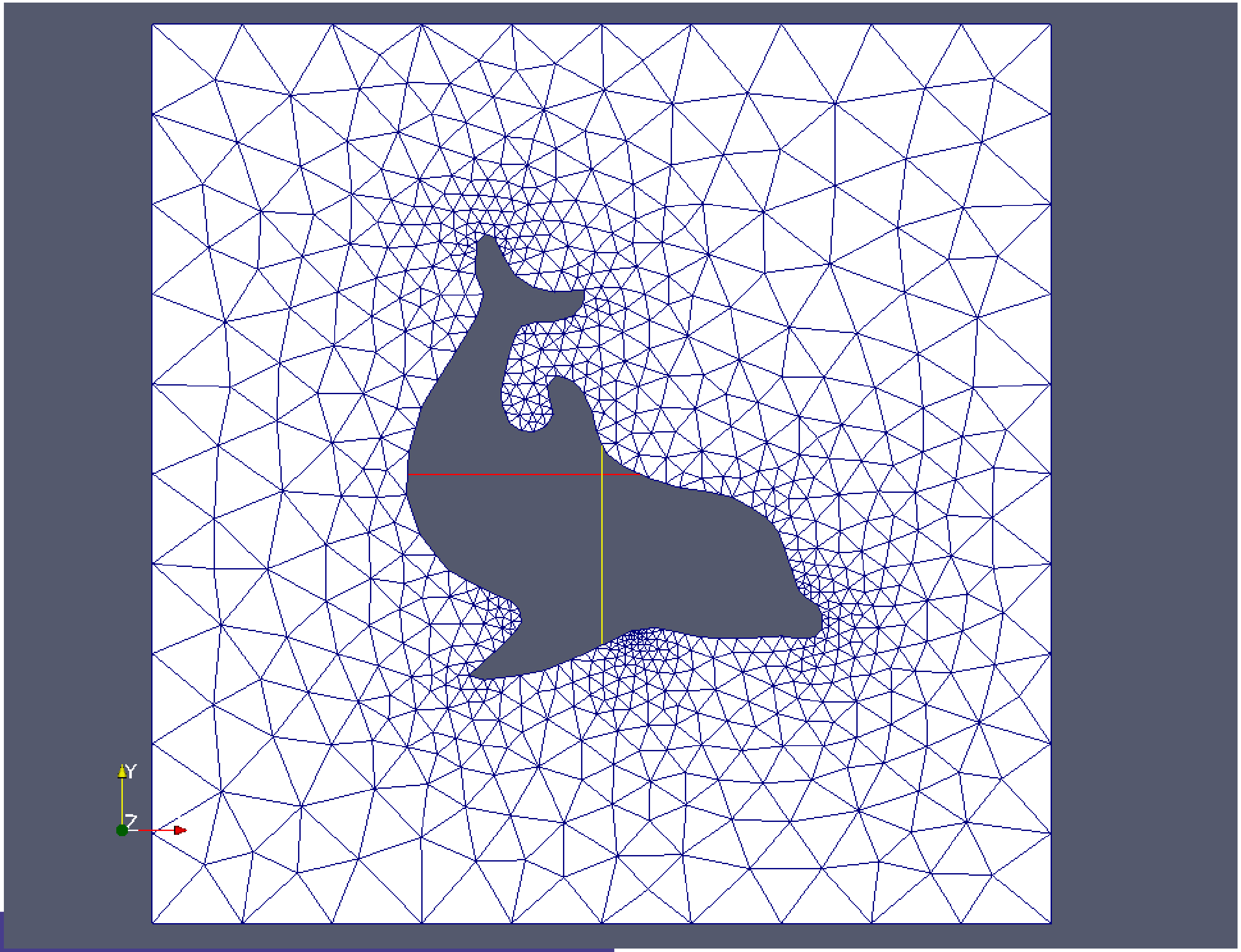
Mesh refinement

Recursive bisection (Rivara)

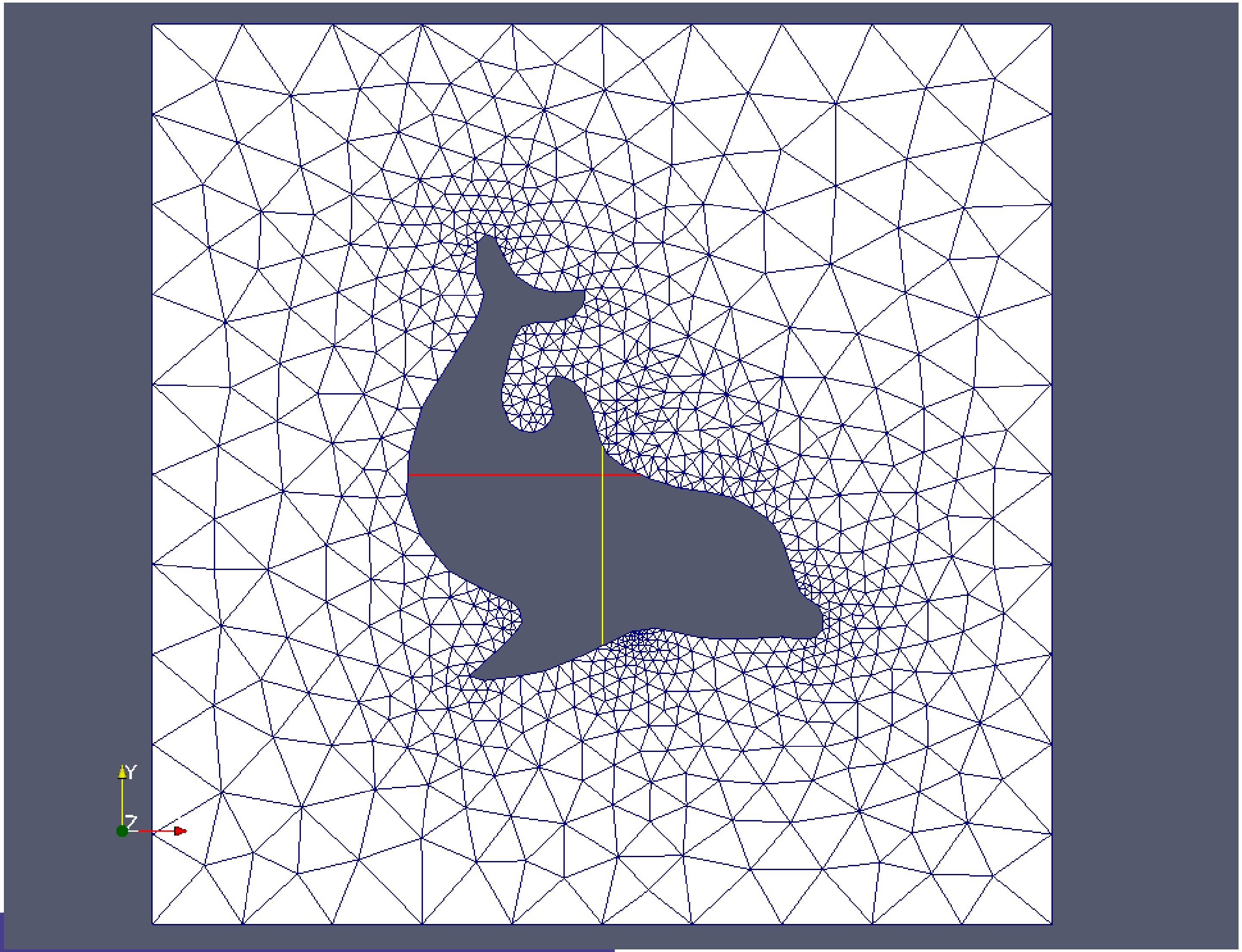
function bisect(K):
  Split longest edge e
  While $K_i(e)$ is non-conforming
    bisect(K_i)
where $K_i(e)$ is cell incident on edge e.

Same algorithm in 2D/3D. Bound on cell quality (smallest angle).

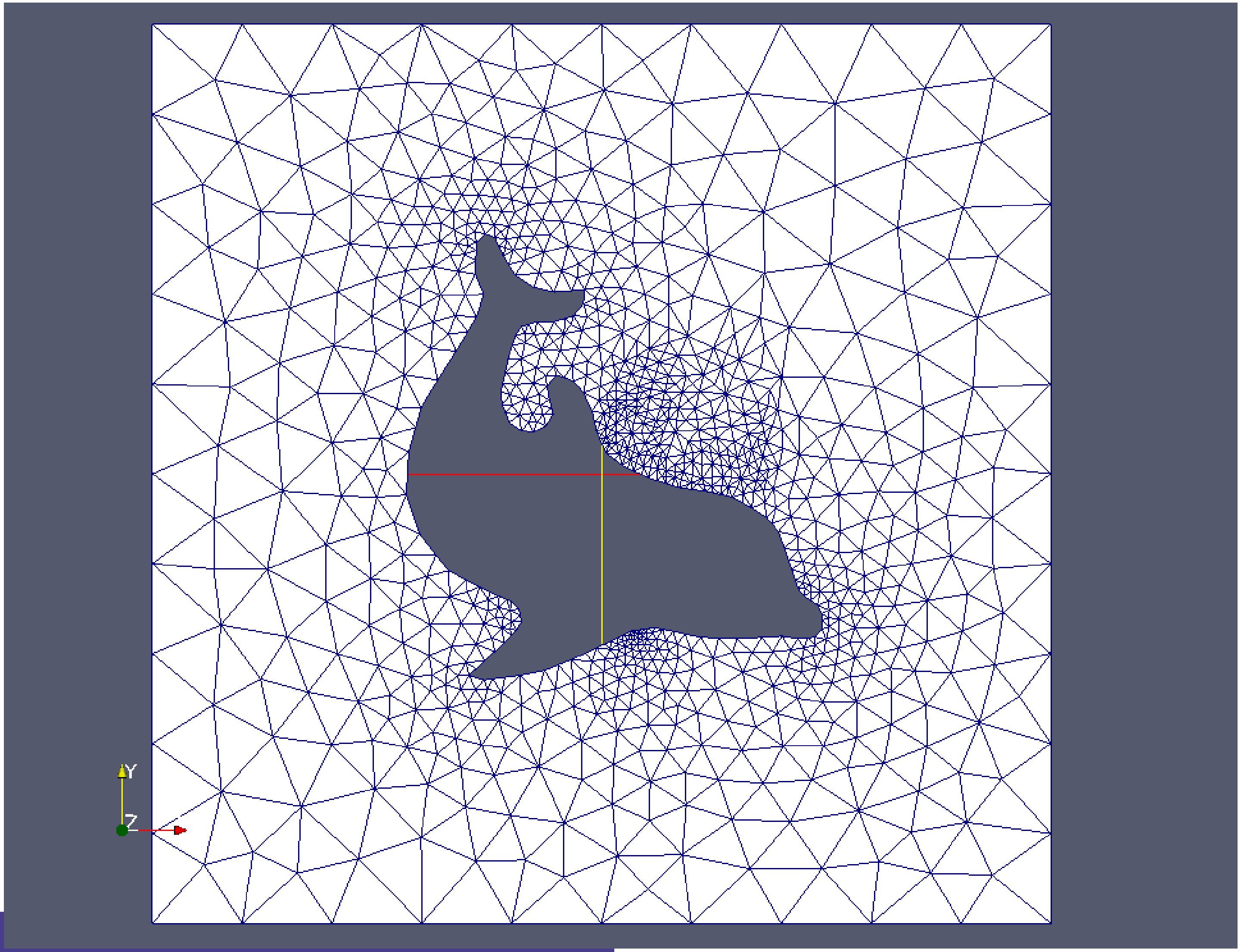
Mesh refinement



Mesh refinement



Mesh refinement



Appendix:

A posteriori/duality 2D/3D

We have *primal equation*:

$-\nabla \cdot (a\nabla u) - f = 0$

We introduce *dual equation*:

$-\nabla \cdot (a\nabla \phi) - \psi = 0$

which is defined by:

$(Av, w) = (v, A^* w)$

where A is the diff. op. for the primal eq. and A^* for the dual eq.
(in this case they are the same).
(Homogenous Dirichlet BC)

Duality 2D

We compute solution U by Galerkin's method:
$\int_\Omega -a\nabla U \cdot \nabla v - fvdx = 0, \quad \forall v \in V_h, \quad U \in V_h$

Error $e = u - U$ satisfies:
$\int_\Omega a\nabla e \cdot \nabla wdx = \int_\Omega a\nabla U \cdot \nabla w - fwdx$

Duality 2D

Want to bound quantity $M(e) = (e, \psi)$:

$$(e,\psi) = \int_\Omega e\psi dx = \int_\Omega e(-\nabla \cdot (a\nabla\phi))dx =$$

$$\int_\Omega a\nabla e \nabla\phi dx + \int_\Gamma e(\nabla\phi)\cdot n]ds = [e = 0, x \in \Gamma] =$$

$$\int_\Omega a\nabla U \cdot \nabla\phi - f\phi dx =$$

$$[GO] = \int_\Omega -a\nabla U \cdot \nabla(\phi - \pi\phi) + f(\phi - \pi\phi)dx =$$

$$\sum_{K_i} \int_{K_i} (\nabla(a\nabla U) + f)(\phi - \pi\phi)dx + \int_{\partial K_i} a(\nabla U)\cdot n(\phi - \pi\phi)ds$$

Duality 2D

We get facet (∂K_i) integrals from both cells sharing the facet F. We write the sum (normals are opposite, so they have opposite signs):

$$\int_F [a(\nabla U) \cdot n](\phi - \pi\phi) ds$$

Duality 2D

Two ways to continue: use interpolation estimate for boundary expression or express boundary integral as interior integral:

$$\int_F [a(\nabla U) \cdot n](\phi - \pi\phi) ds =$$

$$\int_F h^{-1}[a(\nabla U) \cdot n](\phi - \pi\phi) h ds \approx$$

$$\int_F h^{-1}[a(\nabla U) \cdot n](\phi - \pi\phi) dx$$

We can then continue with an interior integral:

$$\sum_{K_i} \int_{K_i} (\nabla(a\nabla U) + f + h^{-1}[a(\nabla U) \cdot n)])(\phi - \pi\phi) dx$$

Duality 2D

Where we can now just like for 1D use Cauchy-Schwartz and an interpolation estimate:

$$|(e,\psi)| = \sum_{K_i} \int_{K_i} (\nabla(a\nabla U) + f + h^{-1}[a(\nabla U)\cdot n)])(\phi - \pi\phi)dx \leq$$

$$\|\hat{R}(U)\|\|\phi - \pi\phi\| \leq h^2 \|\hat{R}(U)\|\|D^2\phi\|$$

where we identify $S = \|D^2\phi\|$ as a *stability factor*.
and $\hat{R}(U) = \nabla(a\nabla U) + f + h^{-1}[a(\nabla U)\cdot n)]$ piecewise constant (constant on each cell).