Courses/FEM/modules/science

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Contents

Differential equations

Science is based on modeling reality, mainly with differential equations (http://en.wikipedia.org/wiki/Differential_equation) .

Precondition

No precondition needed.

Theory

Partial differential equations

An abstract form

$$A(u(x)) = f, \quad x \in \Omega$$

where:

- $u(x)$ is the unknown **solution**
- A is a **differential operator**.
- Ω is the **domain**, i.e. $\Omega = [0, 1]$.

- f is a given **source term**.
- typically the x is dropped, i.e. $u = u(x)$

Initial value problem

$$u(x_0) = g$$

Here x is typically a "time" variable.

Boundary value problem

$$u(x) = g, \quad x \in \Gamma$$

Here x is typically a "space" variable.

Initial boundary value problem

Can have both at the same time.

Residual

We define the **residual** function $R(U)$ as

$$R(U) = A(U) - f$$

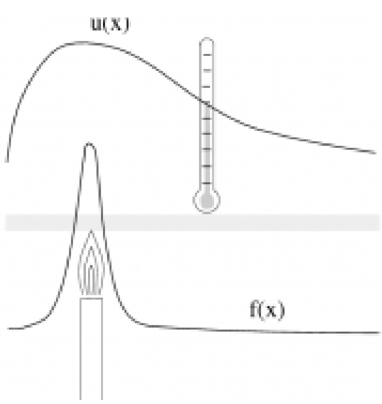
We can thus define an **equation** as computing an object u such that

$$R(u) = 0$$

Poisson's equation

A model for stationary heat conduction

We model heat conduction a thin heat-conducting wire occupying the interval $[0, 1]$ that is heated by a *heat source* of intensity $f(x)$.



We are interested in the stationary distribution of the temperature $u(x)$ in the wire. We let $q(x)$ denote the heat flux in the direction of the positive x-axis in the wire at $0 < x < 1$. Conservation of energy in a stationary case requires that the net heat flux through the endpoints of an arbitrary sub-interval (x_1, x_2) of $(0, 1)$ be equal to the heat produced in (x_1, x_2) per unit time:

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x)\, dx.$$

By the Fundamental Theorem of Calculus,

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} q'(x)\, dx,$$

from which we conclude that

$$\int_{x_1}^{x_2} q'(x)\, dx = \int_{x_1}^{x_2} f(x)\, dx.$$

Since x_1 and x_2 are arbitrary, assuming that the integrands are continuous, we conclude that

$$q'(x) = f(x) \quad \text{for } 0 < x < 1,$$

which expresses conservation of energy in differential equation form. We need an additional equation that relates the heat flux q to the temperature gradient (derivative) u' called a *constitutive equation*. The simplest constitutive equation for heat flow is *Fourier's law*:

$$q(x) = -a(x)u'(x),$$

which states that heat flows from warm regions to cold regions at a rate proportional to the temperature gradient $u'(x)$. The constant of proportionality is the *coefficient of heat conductivity* $a(x)$, which we assume to be a positive function in $[0, 1]$. Combining energy conservation and Fourier's law gives the *stationary heat equation* in one dimension:

$$-(a(x)u'(x))' = f(x) \quad \text{for } 0 < x < 1.$$

To define a solution u uniquely, the differential equation is complemented by *boundary conditions* imposed at the boundaries $x = 0$ and $x = 1$. A common example is the homogeneous *Dirichlet* conditions $u(0) = u(1) = 0$, corresponding to keeping the temperature zero at the endpoints of the wire. The result is a *two-point boundary value problem*:

$$\begin{aligned} -(au')' &= f, \quad x \in (0, 1) \\ u(0) &= u(1) = 0 \end{aligned}$$

The boundary condition $u(0) = 0$ may be replaced by $-a(0)u'(0) = q(0) = 0$, corresponding to prescribing zero heat flux, or insulating the wire, at $x = 0$. Later, we also consider non-homogeneous boundary conditions of the from $u(0) = u_0$ or $q(0) = g$ where u_0 and g may be different from zero.

Software

Familiarize yourself the software tools and the computer environment for the course.

Postcondition

You should now be familiar with:

- why it's important to be able to solve differential equations
- the notation for differential equations
- some basic differential equations used in science:
    - Poisson's equation
    - The convection equation
    - The wave equation
- boundary conditions (Dirichlet, Neumann, Robin)

Exercises

CDE 6.7:

Determine the solution u of the stationary heat equation with $a(x) = 1$ by symbolic computation by hand in the case $f(x) = 1$ and $f(x) = x$.

Examination

1.1.

Investigate a different science domain (**elasticity**, **fluid flow** or **electromagnetism**) and make your own derivation of Poisson's equation (in 1D or 2D) for that domain using appropriate simplifications. Discuss the physical interpretation of boundary conditions.

1.2.

Solve Poisson's equation in 2D using FEniCS with data of your choice. Specify homogenous Neumann boundary conditions on one part of the boundary and homogenous Dirichlet boundary conditions on another part of the domain. Plot the results.