Courses/FEM/modules/galerkin

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Galerkin's method

Precondition

- Science
- Function approximation

Theory

Integration by parts

We recall formulas for integration by parts (in index notation)

$$\int_\Omega D_{x_i} vw dx = \int_\Gamma vwn_i ds - \int_\Omega vD_{x_i} w dx, \quad i = 1, 2, \ldots, d$$

where the special case $d = 1$ is

$$\int_a^b D_x vw dx = [vw]_a^b - \int_a^b vD_x w dx$$

Replacing $w = D_{x_i} w$ gives (repeated indices in a term implies summation)

$$\sum_{i=1}^{2} \int_{\Omega} D_{x_i} v D_{x_i} w dx = \sum_{i=1}^{2} \int_{\Gamma} v D_{x_i} w n_i ds - \int_{\Omega} v D_{x_i} D_{x_i} w dx$$

which in operator form is

$$\int_{\Omega} \nabla v \cdot \nabla w dx = \int_{\Gamma} v(\nabla w \cdot n) ds - \int_{\Omega} v \Delta w dx$$

where we have the normal $n = (n_1, \ldots, n_d)$.

Galerkin's method

We seek to construct an approximate solution of the form $U = \Sigma_i^N \xi_i \phi_i \in V_h$ to a differential equation with an exact solution $u \in V$. We choose Poisson's equation as a model problem

$$\begin{aligned} R(u) = \Delta u - f = 0, \quad x \in \Omega \\ u = 0, \quad x \in \Gamma \end{aligned}$$

However, since ΔU and f typically belong to different function spaces, the residual $R(U)$ can in general not be zero. The best we can hope for is that $R(U)$ is orthogonal to V_h, which means solving the equation

$$\begin{aligned} (R(U), v) = (\Delta U - f, v) = 0, \quad x \in \Omega, \quad \forall v \in V_h \\ U = 0, \quad x \in \Gamma \end{aligned}$$

This property of U is the *Galerkin orthogonality*.

The true solution u satisfies a stronger orthogonality condition

$$\begin{aligned} (R(u), v) = (\Delta u - f, v) = 0, \quad x \in \Omega, \quad \forall v \in V \\ u = 0, \quad x \in \Gamma \end{aligned}$$

This is called the *weak form* of the equation, where we test against all functions, not just those in our approximation space. It is useful to start from the weak form, perform transformations such as integration by parts, then plug in U and choose to only test against the subset $V_h \in V$.

When V_h contains only piecewise linear functions we cannot directly plug in U (since U then does not have second derivatives). We use integration by parts to move one derivative to the test function

$$\begin{aligned} (\Delta u - f, v) = 0, \quad x \in \Omega, \quad \forall v \in V \quad \Rightarrow \\ -(\nabla u, \nabla v) + \int_{\Gamma} v(\nabla u \cdot n) ds - (f, v) = 0, \quad x \in \Omega, \quad \forall v \in V \\ u = 0, \quad x \in \Gamma \end{aligned}$$

Boundary conditions

$$R(u) = -(au')' - f$$

$$(R(u), v) = \int_0^1 -(au')'v - fvdx = [w = au'] =$$

$$\int_0^1 (au')v' - fvdx + au'(1)v(1) - au'(0)v(0)$$

For homogenous Dirichlet BC we can use $v(a) = v(b) = 0$ (the boundary condition is enforced in the function space V).

For homogenous Neumann BC we have $-au' = 0$

Software

Construct finite element solutions to differential equations and verify the Galerkin orthogonality.

Postcondition

You should now be familiar with:

- Galerkin orthogonality
- Weak forms
- Linear and bilinear forms
- Integration by parts
- Energy norm
- Finite element basis functions

Exercises

- CDE: 6.2, 6.3, 6.8, 6.9, 6.10, 6.11, 8.6, 8.11, 8.12, 8.13, 13.30, 15.14, 15.19, 15.20, 15.21

Examination

1.1

Derive the weak form of the equation

$$R(u) = \alpha u - \nabla \cdot (\epsilon \nabla u) = 0$$

with boundary conditions of your choice (you can assume homogenous Dirichlet for simplicity) for the purpose of discretization with piecewise linear polynomials.

1.2

Give an explanation of the Galerkin orthogonality in your own language. What is the connection with (orthogonal) L2 projection?

1.3

Using FEniCS:

Compute a solution of Poisson's equation in 2D (you may use a previously computed solution). Verify the Galerkin orthogonality with functions from the approximation space V_h. Note: remember that we assume that functions in the test space are zero on the part of the boundary where we specify Dirichlet boundary conditions. Hint: you already have a function which is in V_h..

[TODO]

- Define energy norm