Courses/FEM/modules/assembly

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Contents

Assembly of discrete systems

Precondition

- Science
- Function approximation
- Galerkin's method

Theory

The Galerkin finite element method

We consider Poisson's equation in the case $a \equiv 1$, that is

$$-u'' = f, \quad x \in (0,1)$$
$$u(0) = u(1) = 0$$

and formulate the simplest finite element method for the boundary value problem based on continuous piecewise linear approximation.

We let $\mathcal{T}_h : 0 = x_0 < x_1 < \ldots < x_{M+1} = 1$, be a *partition* or (*triangulation*) of $I = (0, 1)$ into subintervals $I_j = (x_{j-1}, x_j)$ of length $h_j = x_j - x_{j-1}$ and let $V_h = V_h^{(1)}$ denote the set of continuous piecewise linear functions on $\mathcal{T}_h$ that are zero at $x = 0$ and $x = 1$.

We have seen that V_h is a finite dimensional vector space of dimension M with a basis consisting of the hat functions $\{\phi_j\}_{j=1}^{M}$ illustrated in figure. The coordinates of a function v in V_h in this basis are the values $v(x_j)$ at the interior nodes x_j, $j = 1, \ldots, M$, and a function $v \in V_h$ can be written

$$v(x) = \sum_{j=1}^{M} v(x_j)\phi_j(x).$$

Note that because $v \in V_h$ is zero at 0 and 1, we do not include ϕ_0 and ϕ_{M+1} in the set of basis functions for V_h.

As in the previous example, Galerkin's method is based on stating the differential equation $-u'' = f$ in the form

$$\int_0^1 (-u'' - f)v\,dx = 0 \quad \text{for all functions} v,$$

corresponding to the residual $-u'' - f$ being orthogonal to the test functions v. However, since the functions in V_h do not have second derivatives, we can't simply plug a candidate for an approximation of u in the space V_h directly into this equation. To get around this technical difficulty, we use integration by parts to move one derivative from u'' onto v assuming v is differentiable and $v(0) = v(1) = 0$:

$$-\int_0^1 u''v\,dx = -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v'\,dx = \int_0^1 u'v'\,dx,$$

where we used the boundary conditions on v. We are thus led to the following *variational formulation* of bvp: find the function u with $u(0) = u(1) = 0$ such that

$$\int_0^1 u'v'\,dx = \int_0^1 fv\,dx,$$

for all functions v such that $v(0) = v(1) = 0$. We also refer to this as a *weak form*.

The Galerkin finite element method for the boundary value problem is the following finite-dimensional analog: find $U \in V_h$ such that

$$\int_0^1 U'v'\,dx = \int_0^1 fv\,dx \quad \text{for all } v \in V_h.$$

We note that the derivatives U' and v' of the functions U and $v \in V_h$ are piecewise constant functions. and are not defined at the nodes x_i. However, the integral with integrand $U'v'$ is nevertheless uniquely defined as the sum of integrals over the sub-intervals. This is due to the basic fact of integration that two functions that are equal except at a finite number of points, have the same integral.

By the same token, the value (or lack of value) of U' and v' at the distinct node points x_i does not affect the value of $\int_0^1 U'v'\,dx$.

The equation tested against the space V expresses the fact that the residual error $-u'' - f$ of the exact solution is orthogonal to *all* test functions v. Similarly, the finite element formulation is a way of forcing in weak form the residual error of the finite element solution U to be orthogonal to the finite dimensional set of test functions v in V_h.

The discrete system of equations

Using the basis of hat functions $\{\phi_j\}_{j=1}^M$, we have

$$U(x) = \sum_{j=1}^{M} \xi_j \phi_j(x)$$

and determine the nodal values $xi_j = U(x_j)$ using the Galerkin orthogonality bvpfem. Substituting, we get

$$\sum_{j=1}^{M} \xi_j \int_0^1 \phi_j' v'\,dx = \int_0^1 fv\,dx,$$

for all $v \in V_h$. It suffices to check bvpfemsub for the basis functions $\{\phi_i\}_{i=1}^M$, which gives the $M \times M$ linear system of equations

$$\sum_{j=1}^{M} \xi_j \int_0^1 \phi_j' \phi_i'\,dx = \int_0^1 f\phi_i\,dx, \quad i = 1, \dots, M,$$

for the unknown coefficients $\{\xi_j\}$. We let $\xi = (\xi_j)$ denote the vector of unknown coefficients and define the $M \times M$ *stiffness matrix* $A = (a_{ij})$ with coefficients

$$a_{ij} = \int_0^1 \phi_j' \phi_i'\,dx,$$

and the *load vector* $b = (b_i)$ with

$$b_i = \int_0^1 f\phi_i \, dx.$$

These names originate from early applications of the finite element method in structural mechanics. Using this notation, bvpfemeqn is equivalent to the linear system

$$A\xi = b.$$

In order to solve for the coefficients of U, we first have to compute the stiffness matrix A and load vector b. For the stiffness matrix, we note that a_{ij} is zero unless $i = j - 1$, $i = j$, or $i = j + 1$ because otherwise either $\phi_i(x)$ or $\phi_j(x)$ is zero on each sub-interval occurring in the integration. We illustrate this in threehat.

We compute a_{ii} first. Using the definition of the ϕ_i,

$$\begin{aligned} \phi_i(x) = (x - x_{i-1})/h_i, &\quad x_{i-1} \le x \le x_i, \\ (x_{i+1} - x)/h_{i+1}, &\quad x_i \le x \le x_{i+1}, \end{aligned}$$

and $\phi_i(x) = 0$ elsewhere, the integration breaks down into two integrals:

$$a_{ii} = \int_{x_{i-1}}^{x_i} \Big(\frac{1}{h_i}\Big)^2 \, dx + \int_{x_i}^{x_{i+1}} \Big(\frac{-1}{h_i}\Big)^2 \, dx = \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

since $\phi_i' = 1/h_i$ on (x_{i-1}, x_i) and $\phi_i' = -1/h_{i+1}$ on (x_i, x_{i+1}), and ϕ_i is zero on the rest of the sub-intervals. Similarly,

$$a_{i\,i+1} = \int_{x_i}^{x_{i+1}} \frac{-1}{h_{i+1}} \, \frac{1}{h_{i+1}} \, dx = -\frac{1}{h_{i+1}} \, .$$

Problem

Prove that $a_{i-1\,i} = -1/h_i$ for $i = 2, 3, \ldots, M$.

Problem

Determine the stiffness matrix A in the case of a uniform mesh with meshsize $h_i = h$ for all i.

We compute the coefficients of b in the same way to get

$$b_i = \int_{x_{i-1}}^{x_i} f(x)\,\frac{x - x_{i-1}}{h_i}\,dx + \int_{x_i}^{x_{i+1}} f(x)\,\frac{x_{i+1} - x}{h_{i+1}}\,dx, \quad i = 1, \ldots, M.$$

General assembly algorithm

In general the matrix A_h , representing a bilinear form

$$a(u, v) = (A(u), v),$$

is given by

$$(A_h)_{ij} = a(\varphi_j, \hat{\varphi}_i).$$

and the vector b_h representing a linear form

$$L(v) = (f, v),$$

is given by

$$(b_h)_i = L(\hat{\varphi}_i).$$

Computing $(A_h)_{ij}$

Note that

$$\begin{aligned} (A_h)_{ij} = & \quad a(\varphi_j, \hat{\varphi}_i) = \int_\Omega A(\varphi_j)\hat{\varphi}_i\,dx \\ = & \quad \sum_{K \in \mathcal{T}} \int_K A(\varphi_j)\hat{\varphi}_i\,dx = \sum_{K \in \mathcal{T}} a(\varphi_j, \hat{\varphi}_i)_K. \end{aligned}$$

Iterate over all elements K and for each element K compute the contributions to all $(A_h)_{ij}$, for which φ_j and $\hat{\varphi}_i$ are supported within K .

Assembling A_h

for all elements $K \in \mathcal{T}$

for all test functions $\hat{\varphi}_i$ on K

for all trial functions φ_j on K

1. Compute $I = a(\varphi_j, \hat{\varphi}_i)_K$

2. Add I to $(A_h)_{ij}$

end

end

end

Assembling b

for all elements $K \in \mathcal{T}$

for all test functions $\hat{\varphi}_i$ on K

1. Compute $I = L(\hat{\varphi}_i)_K$

2. Add I to b_i

end

end

Mapping from a reference element - isoparametric mapping

We want to compute basis functions and integrals on a reference element K_0

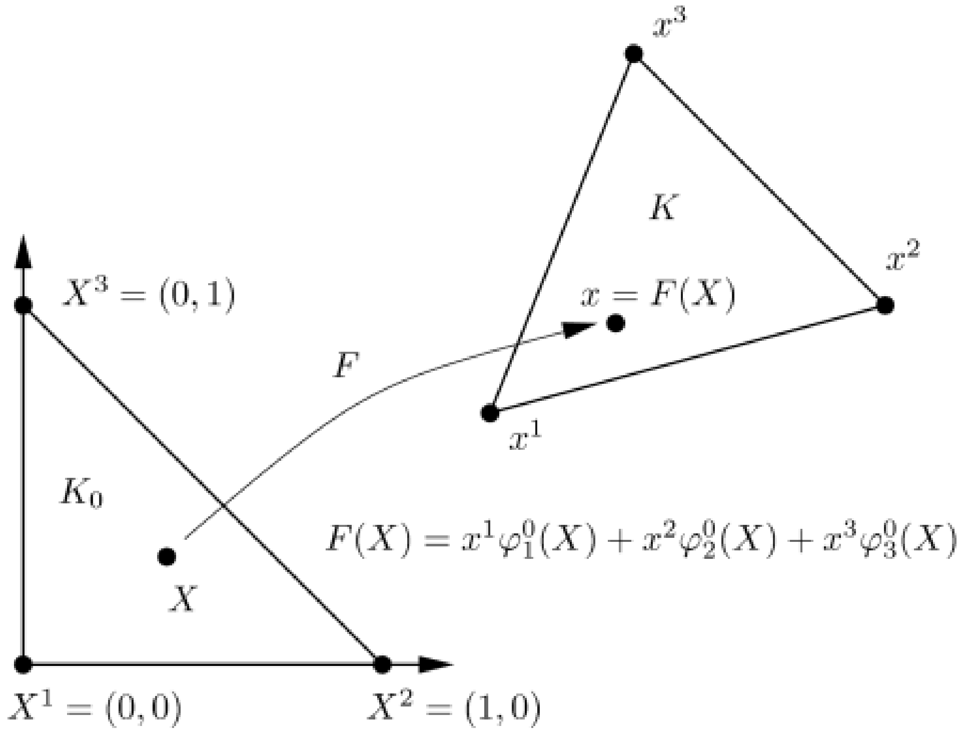
Most common mapping is isoparametric mapping (use the basis functions also to define the geometry):

$$x(X) = F(X) = \sum_{i=1}^{n} \phi_i(X) x_i$$

Linear basis functions $\Rightarrow$ Affine mapping

$$x(X) = F(X) = BX + b$$

The mapping $F : K_0 \rightarrow K$



Some basic calculus

Let $v = v(x)$ be a function defined on a domain Ω and let

$$F : \Omega_0 \to \Omega$$

be a (differentiable) mapping from a domain Ω_0 to Ω . We then have $x = F(X)$ and

$$\begin{aligned}\int_\Omega v(x)\,dx = &\int_{\Omega_0} v(F(X))\,||\det \partial F_i/\partial X_j||\,dX\\ = &\int_{\Omega_0} v(F(X))\,||\det \partial x/\partial X||\,dX.\end{aligned}$$

Affine mapping

When the mapping is affine, the determinant is constant:

$$
\begin{aligned}
& \int_K \varphi_j(x)\hat{\varphi}_i(x)\,dx \\
= & \int_{K_0} \varphi_j(F(X))\hat{\varphi}_i(F(X))\,||\det \partial x/\partial X||\,dX \\
= & ||\det \partial x/\partial X|| \int_{K_0} \varphi_j^0(X)\hat{\varphi}_i^0(X)\,dX
\end{aligned}
$$

Transformation of derivatives

To compute derivatives, we use the transformation

$$
\nabla_X = \left(\frac{\partial x}{\partial X}\right)^\top \nabla_x,
$$

or

$$
\nabla_x = \left(\frac{\partial x}{\partial X}\right)^{-\top} \nabla_X.
$$

The stiffness matrix

For the computation of the stiffness matrix, this means that we have

$$
\begin{aligned}
& \int_K \epsilon(x)\nabla\varphi_j(x)\cdot\nabla\hat{\varphi}_i(x)\,dx \\
& = \int_{K_0} \epsilon_0(X)\left[(\partial x/\partial X)^{-\top}\nabla_X\varphi_j^0(X)\right]\cdot\left[(\partial x/\partial X)^{-\top}\nabla_X\hat{\varphi}_i^0(X)\right]||\det(\partial x/\partial X)||\,dX.
\end{aligned}
$$

Note that we have used the short notation $\nabla = \nabla_x$.

Software

FEniCS implements the above assembly algorithm as the `assemble()` function.

Here is a simple Python implementation of the general assembly algorithm:
http://www.icarusmath.com/icarus/images/Myassemble.zip

Postcondition

You should now be familiar with:

- Mapping from a reference cell
- The general assembly algorithm
- How to implement boundary conditions
- How to construct the discrete system for a linear/nonlinear time-independent PDE with Galerkin's method

Exercises

CDE: 8.11, 8.12, 8.13,8.22, 8.23, 14.9

1.1

(Advanced):

Consider the non-linear equation $R(u) = -\Delta u - u^2 = 0$. Derive the weak form and go through the steps of discretization until you have a discrete (algebraic) equation. What is different from the linear case?

Examination

1.1

In Python (or a language of your choice) or with pen and paper (will probably be quickest):

Compute the integral $(\nabla\phi_0, \nabla\phi_0)$ on the reference triangle (with vertices $X_0 = (0,0), \quad X_1 = (1,0), \quad X_2 = (0,1)$, and thus $\phi_0 = 1 - x - y$). Compute the mapping $F(X)$ to a physical triangle of your choice. Compute the integral on the physical triangle using the above formula for coordinate transform.

1.2

Why is it enough to only test against the functions $\phi_i \in V_h$ and not against all functions $v \in V_h$?

[TODO]

- Add dof mapping