Courses/FEM/modules/existence

From Icarus

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Contents

Existence and uniqueness of solutions

Precondition

- Science
- Function approximation

Theory

Abstract framework for linear elliptic PDE

Function space (Hilbert space) V with norm $||v|_V$ and inner product $(v, w)_V$.

Bilinear form a

$$V \times V \to R$$

determined by underlying differential equation.

Linear form L

$$V \to R$$

determined by data.

Search for solution in function space V, formulate equation using a and L.

Linear form

A linear form $L(v)$ is a function taking values (functions) in V into R

$$L(v) \in R \forall v \in V$$

. $L(v)$ linear in the argument

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2) \\ \forall \alpha_i \in R, v_i, \in V$$

Bilinear form

A bilinear form $a(v, w)$ is a function taking values (functions) in $V \times V$ into R

$$a(v, w) \in R \forall v, w \in V$$

.

$a(v, w)$ linear in each argument

$$a(\alpha_1 v_1 + \alpha_2 v_2, w_1) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_2, w_1) \\ a(v_1, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_2, w_1) \\ \forall \alpha_i \in R, v_i, w_i \in V$$

Existence and uniqueness

Find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V$$

Problem: do such solutions $u \in V$ exist? (Existence)

Problem: if so, is u unique? (Uniqueness)

For certain requirements (a certain class of equations: linear and elliptic) on $a(v, w)$, $L(v)$ and V, we can show existence and uniqueness of solutions u.

Function space

Natural space for second-order elliptic differential equations:

Bounded function and gradient

$$V = \{v : \int_\Omega (|\nabla v|^2 + v^2) dx < C\}$$

Inner product

$$(v, w)_V = \int_\Omega (\nabla v \cdot \nabla w + vw) dx$$

Generates the norm

$$\|v\|_V = \sqrt{\int_\Omega (|\nabla v|^2 + v^2) dx}$$

Usual notation

$$V = H^1(\Omega)$$

where index 1 denotes one derivative (H^2 would add a term with the second derivative). Named after Sobolev.

Lax-Milgram's theorem

If $a(v, w)$ is a continuous, coercive, bilinear form on V and $L(v)$ is a continuous, linear form on V, then there exists unique $u \in V$ satisfying $a(u, v) = L(v), \quad \forall v \in V$.

Form requirements

Coercivity (or V-ellipticity) of $a(v, w)$:

There exists constant $\kappa_1 > 0$

$$a(v, v) \geq \kappa_1 \|v\|_V^2$$

Continuity of $a(v, w)$:

There exists constant $\kappa_2 > 0$

$$|a(v,w)| \leq \kappa_2 \|v\|_V \|w\|_V$$

Continuity of $L(v)$:

There exists constant $\kappa_3 > 0$

$$|L(v)| \leq \kappa_3 \|v\|_V$$

Discussion of continuity (by linearity)

$$|L(v)| \leq \kappa_3 \|v\|_V \Rightarrow |L(v) - L(w)| \leq \kappa_3 \|v - w\|_V$$

Recall Lipschitz continuity for a function f

$$R \to R$$

$$|f(x) - f(y)| \leq L|x - y|$$

In this course we also require that $a(v,w)$ be symmetric when discussing existence/uniqueness.

Poincaré's inequality

$\|v\|^2_{L_2(\Omega)} \leq C(\|v\|^2_{L_2(\Gamma)} + \|\nabla v\|^2_{L_2(\Omega)})$

for $v = 0, x \in \Gamma \Rightarrow \|v\|_{L_2(\Gamma)} = 0$

$$\|v\|_{L_2(\Omega)} \leq C\|\nabla v\|_{L_2(\Omega)}$$

Consequence

$$\|v\|_{L_2(\Omega)} + \|\nabla v\|_{L_2(\Omega)} \leq C_2 \|\nabla v\|_{L_2(\Omega)}$$

Example equation

$$\begin{aligned} -\Delta u + u &= f \\ \nabla u \cdot n &= 0, \quad x \in \Gamma \end{aligned}$$

Linear form

$$L(v) = (f, v)$$

Bilinear form

$$a(u,v) = (\nabla u, \nabla v) + (u,v)$$

Verification of requirements for Lax-Milgram's theorem:

Coercivity

$$a(v,v) = (\nabla v, \nabla v) + (v,v) = \|\nabla v\|^2 + \|v\|^2 = \|v\|_V^2 \Rightarrow$$

$\kappa_1 = 1$

Continuity of a

$$|a(v,w)| = |(\nabla v, \nabla w) + (v,w)| = |(v,w)_V| \leq \|v\|_V \|w\|_V \Rightarrow$$

$\kappa_2 = 1$

Continuity of L

$$|L(v)| = |(f,v)| \leq \|f\|_{L_2} \|v\|_{L_2} \leq \|f\|_{L_2} \|v\|_V \Rightarrow$$

$\kappa_3 = \|f\|_{L_2}$

where we "for free" see the requirement that $\|f\|_{L_2}$ must be bounded to satisfy the requirements.

Software

Postcondition

You should now be familiar with:

- Lax-Milgram's theorem
- Poincaré's inequality
- Standard function space V and norm
- How to verify uniqueness and existence for a linear elliptic PDE

Exercises

See chapter 21 in CDE.

Examination

1.1.

Show that the equation

$$\begin{aligned} -\Delta u = f, &\quad x \in \Omega \\ u = 0, &\quad x \in \Gamma \end{aligned}$$

satisfies the Lax-Milgram theorem (has a unique solution).

1.2.

The equation

$$\begin{aligned} -\Delta u = f, &\quad x \in \Omega \\ (\nabla u) \cdot n = 0, &\quad x \in \Gamma \end{aligned}$$

does not satisfy the Lax-Milgram theorem (may not have a unique solution). Construct an agument showing why not.

[TODO]

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