DN2660 The Finite Element Method: Written Examination
Thursday 2012-10-18, 8-13
Coordinator: Johan Jansson

Aids: none Time: 5 hours

Answers must be given in English. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be given zero points, while a good explanation with an incorrect answer can give some points. Maximum is 30 points.

Good luck,
Johan

Problem 1 - Galerkin's method

Consider the equation:

$$
\begin{aligned}
-\nabla \cdot (a(x)\nabla u(x)) + b(x) \cdot \nabla u(x) = f(x), &\quad x \in \Omega \\
u(x) = 0, &\quad x \in \Gamma
\end{aligned}
$$

where $a(x), b(x)$ and $f(x)$ are known coefficients and $b(x)$ is vector-valued.

Recall the formula:

$$
\int_\Omega D_{x_i} vw dx = \int_\Gamma vwn_i ds - \int_\Omega vD_{x_i} w dx, \quad i = 1, 2, ..., d
$$

1. (2p) Formulate a finite element method (Galerkin's method) for the equation using piecewise linear approximation (cG(1)).
2. (1p) Explain what the Galerkin orthogonality means, both in general and for this equation.
3. (1p) Formulate a Robin boundary condition and use it to enforce the homogenous Dirichlet boundary condition.
4. (2p) What is an L_2 projection? What is the relation to Galerkin's method?

Problem 2 - Stability

Consider the heat equation with zero source:

$$\begin{aligned} \dot{u} - \Delta u = 0, &\quad x \in \Omega, \quad t \in [0, T] \\ u(0, x) = u_0(x) & \\ u(t, x) = 0, &\quad x \in \Gamma \end{aligned}$$

The $cG(1)dG(0)$ method for this equation is:

$$(U_n, v) = (U_{n-1}, v) - k_n(\nabla U(t_n, x), \nabla v)), \quad \forall v \in V_h \times W_k$$

1. (2p) Derive the stability estimate:

   $$\|U_n\| \leq \|U_{n-1}\|$$

   Explain what a stability estimate is in general, and give an interpretation what this particular stability estimate says about the discrete temperature U.

2. (2p) Explain the basic concept behind a streamline diffusion stabilized finite element method.

Problem 3 - Assembly of a linear system

1. (3p) Formulate a general assembly algorithm of a linear system given a bilinear form $a(u, v)$ and linear form $L(v)$ representing a linear boundary value partial differential equation (PDE) in 2D/3D, with a piecewise linear Galerkin approximation (cG(1)). Include explanations of the following concepts:

   - Mesh
   - Map from reference cell
   - Formula for computation of a matrix and vector element
   - Quadrature

2. (2p) Define a basic linear boundary value PDE in 1D or 2D. Apply Galerkin's method, construct a simple mesh and compute a matrix element by hand (you don't have to use a general assembly algorithm here).

Problem 4 - Error estimation

Consider the equation:

$$\begin{aligned} -u'' + u = f, &\quad x \in [0, 1] \\ u(0) = u(1) = 0 & \end{aligned}$$

1. (2p) Show that Galerkin's method is optimal for the equation and derive an a priori error estimate in the energy norm $\|w\|_E$.

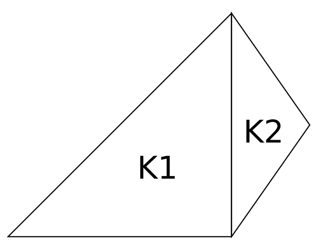
2. (1p) Explain what an a posteriori error estimate is, give a definition of the energy norm for the equation and explain why the energy norm is often used.

3. (3p) Derive an a posteriori error estimate using duality for a general quantity of the error (e, ψ) in the form: $|(e, \psi)| \leq C_i h^2 \|R(U)\| \|\phi''\|$, where ϕ is the dual solution and $R(U)$ the residual. Use continuous piecewise linear approximation and the interpolation estimate $\|\phi - \pi\phi\| \leq C_i h^2 \|\phi''\|$.

Problem 5 - Adaptivity

1. (3p) Formulate an adaptive finite element method based on an a posteriori error estimate with local mesh refinement given a tolerance TOL on a quantity or norm of the error $e = u - U$. Discuss why adaptivity is important.

2. (2p) Formulate the Rivara recursive bisection algorithm. Consider the mesh:



Mark the triangle K2 for refinement and perform the Rivara algorithm by hand, show all steps.

Problem 6 - Abstract formulation

1. (2p) Explain what the Lax-Milgram theorem says, what it requires to be satisfied, and what it can be used for.

2. (2p) Define a linear, time-independent boundary value PDE of your choice and show why or why not the Lax-Milgram theorem is satisfied (an argument is sufficient to show that it's not satisfied).