FEM12 - lecture 3

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Outline

- Modules (deadline)
- Repetition
- Boundary conditions
- General assembly algorithm: reference element mapping

Boundary conditions

Essential BC:

- Homogenous Dirichlet BC: $u(0) = 0$
  Enforce in function space:
  $V = \left\{ v : \int_0^1 v^2\,dx < C,\ \int_0^1 (v')^2\,dx < C,\ v(0) = 0 \right\}$

Natural BC:

- Neumann BC: $-a(0)u'(0) = g_N$
- Robin BC: $-a(0)u'(0) + \gamma(u(0) - g_D) = g_N$
  γ is a penalty parameter, with $\gamma = 0 \Rightarrow$ Neumann and $\frac{1}{\gamma} = 0 \Rightarrow$ Dirichlet

Enforce in weak form:
$\int_0^1 (au')v' - fv dx + au'(1)v(1) - au'(0)v(0) = 0$

Computer demonstration

Mapping from a reference element

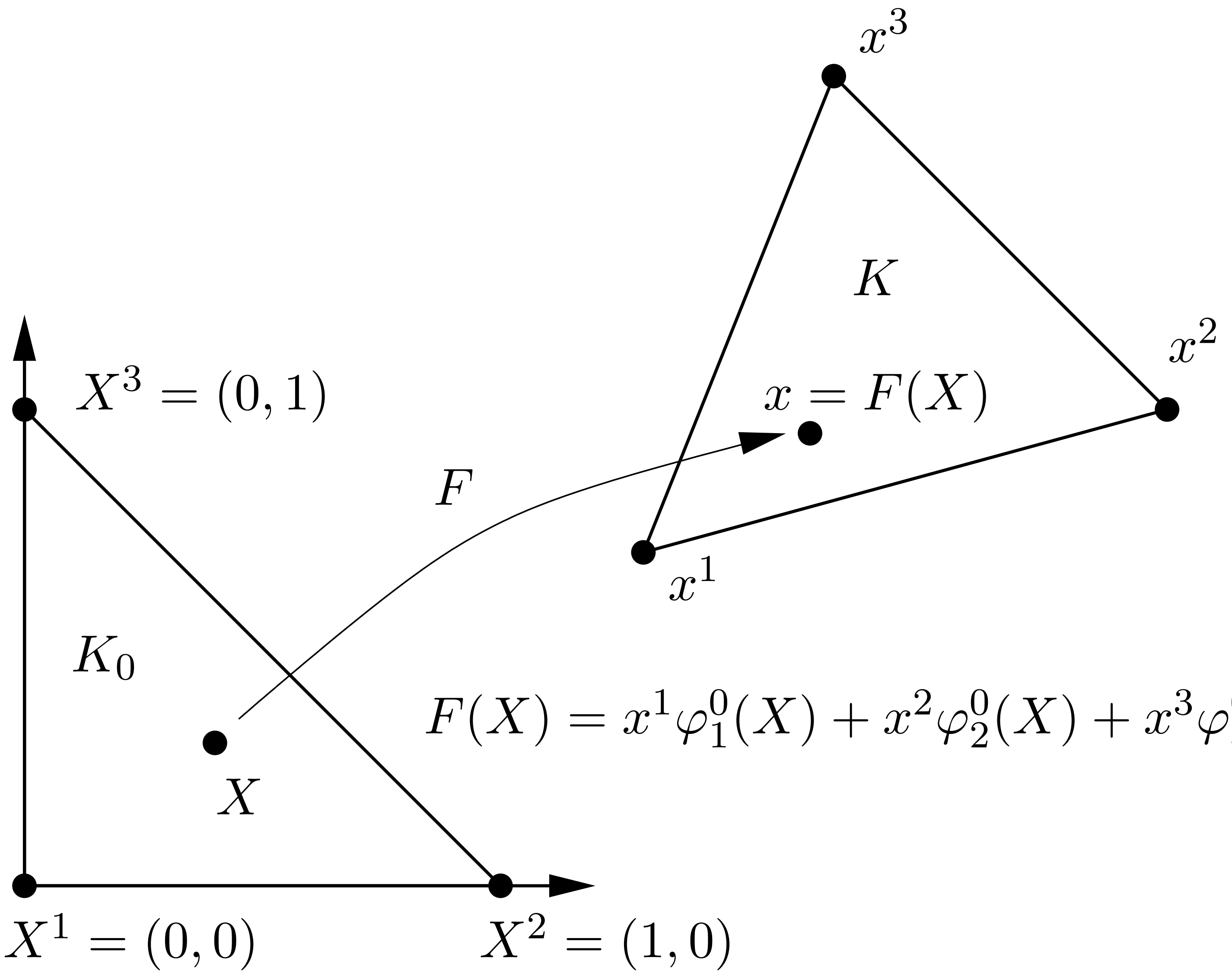
Isoparametric mapping

- We want to compute basis functions and integrals on a *r*eference element K_0
- Most common mapping is isoparametric mapping (use the basis functions also to define the geometry):

$$x(X) = F(X) = \sum_{i=1}^{n} \phi_i(X) x_i$$

- Linear basis functions $\Rightarrow$
Affine mapping: $x(X) = F(X) = BX + b$

The mapping $F : K_0 \to K$



$$F(X) = x^1\varphi_1^0(X) + x^2\varphi_2^0(X) + x^3\varphi_3^0(X)$$

Integration: coordinate transform

Let $v = v(x)$ be a function defined on a domain Ω and let

$$F : \Omega_0 \rightarrow \Omega$$

be a (differentiable) mapping from a domain Ω_0 to Ω. We then have $x = F(X)$ and

$$\begin{aligned}\int_\Omega v(x)\, dx &= \int_{\Omega_0} v(F(X))\, |\det \partial F_i / \partial X_j|\, dX \\ &= \int_{\Omega_0} v(F(X))\, |\det \partial x / \partial X|\, dX.\end{aligned}$$

Affine mapping

When the mapping is affine, the determinant is constant:

$$\begin{aligned} & \int_K \varphi_j(x)\hat{\varphi}_i(x)\, dx \\ = \;& \int_{K_0} \varphi_j(F(X))\hat{\varphi}_i(F(X))\, |\det \partial x/\partial X|\, dX \\ = \;& |\det \partial x/\partial X| \int_{K_0} \varphi_j^0(X)\hat{\varphi}_i^0(X)\, dX \end{aligned}$$

Transformation of derivatives

To compute derivatives, we use the transformation

$$\nabla_X = \left(\frac{\partial x}{\partial X}\right)^\top \nabla_x,$$

or

$$\nabla_x = \left(\frac{\partial x}{\partial X}\right)^{-\top} \nabla_X.$$

The stiffness matrix

For the computation of the stiffness matrix, this means that we have

$$\begin{aligned} & \int_K \epsilon(x) \nabla \varphi_j(x) \cdot \nabla \hat{\varphi}_i(x)\, dx \\ = \quad & \int_{K_0} \epsilon_0(X) \left[(\partial x / \partial X)^{-\top} \nabla_X \varphi_j^0(X) \right] \cdot \left[(\partial x / \partial X)^{-\top} \nabla_X \hat{\varphi}_i^0(X) \right] \cdots \\ & \cdots |\det(\partial x / \partial X)|\, dX. \end{aligned}$$

Note that we have used the short notation $\nabla = \nabla_x$.

in the affine case the $\partial x / \partial X$ are simply elements of the matrix B in $x(X) = F(X) = BX + b$

Computing integrals on K_0

- The integrals on K_0 can be computed symbolically or by quadrature.
- In some cases quadrature is the only option.
- Note that basis functions and products of basis functions can be integrated exactly with quadrature (if polynomial)

Standard form:

$$\int_{K_0} v(X)\, dX \approx |K_0| \sum_{i=1}^{n} w_i v(X^i)$$

where $\{w_i\}_{i=1}^n$ are quadrature weights and $\{X^i\}_{i=1}^n$ are quadrature points in K_0.

Integration/Quadrature in 1D

Midpoint rule

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m+1} f(\frac{x_{i-1}+x_i}{2})h_i + E(f)$$

$$|E(f)| \leq \sum_{i=1}^{m+1} \frac{1}{12} h_i^2 \max_{[x_{i-1},x_i]} |f''| h_i.$$

In other words: the rule can integrate linear polynomials exactly.

Possible to generate quadrature rules for any order of (polynomial) accuracy.

Integration/Quadrature in 2D

Midpoint rule

$$\int_K f(x)\,dx = \sum_{1\le i<j\le 3} f\left(a_K^{ij}\right)\frac{|K|}{3} + E(f)$$

$$|E(f)| \le \sum_{|\alpha|=3} Ch_K^3 \int_K |D^\alpha f(x)|\,dx$$

In other words: the rule can integrate quadratic polynomials in 2D exactly.

Possible to generate quadrature rules for any order of (polynomial) accuracy in 2D/3D as well.