FEM12 - lecture 7

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Contents

FEM for ODE

In general:

$\dot{u} = f(t, u)$

Model problem:

$\dot{u} + au = f(t)$

FEM for ODE: dG(0)

dG(0): $U \in W_k$ with $U_0^- = u_0$, W_k space of piecewise constants

$$\int_{t_{n-1}}^{t_n} \dot{U}v + aUv - fv dt + (U_n - U_{n-1})v = 0, \quad \forall v \in W_k \Rightarrow$$

$$U_n = U_{n-1} - \int_{t_{n-1}}^{t_n} a(t)U(t) + f(t)dt$$

$$U = \sum_{i=1}^{N} \xi_i \phi_i, \quad \phi_i = 1, \quad k_n = t_n - t_{n-1} \Rightarrow$$

$$\xi_n = \xi_{n-1} - \int_{t_{n-1}}^{t_n} a(t_n)\xi_n\phi_n + f(t_n)dt$$

Note that $\dot{U} = 0$ on a time interval $[t_{n-1}, t_n]$

FEM for ODE: dG(0)

Right-point quadrature, we get backward Euler (+ quad. err.):

$$\xi_n = \xi_{n-1} - k_n(a(t_n)\xi_n + f(t_n)) + E_q$$

FEM for ODE: cG(1)

cG(1): $U \in V_k$ with $U(0) = u_0$, V_k space of piecewise linears

$$\int_{t_{n-1}}^{t_n} \dot{U}v + aUv - fv dt = 0, \quad \forall v \in W_k \Rightarrow$$

$$U_n = U_{n-1} - \int_{t_{n-1}}^{t_n} a(t)U(t) + f(t)dt$$

$$U = \sum_{i=1}^{N} \xi_i \phi_i, \quad \phi_i = \frac{t - t_{i-1}}{t_i - t_{i-1}}, \quad k_n = t_n - t_{n-1} \Rightarrow$$

$$\xi_n = \xi_{n-1} - \int_{t_{n-1}}^{t_n} a(t_n)(\xi_{n-1}\phi_{n-1} + \xi_n \phi_n) + f(t_n)dt$$

FEM for ODE: cG(1)

Trapezoid quadrature, we get Crank-Nicolson (+ quad. err.):

$$\xi_n = \xi_{n-1} - \frac{1}{2}k_n(a(t_{n-1})\xi_{n-1} + a(t_n)\xi_n + f(t_{n-1}) + f(t_n)) + E_q$$

Error estimate for dG(0)

For general ODE: $R(u) = \dot{u} - f(t,u) = 0$
Similar construction as for error estimates in space.

$$|u(T) - U(T)| \leq k|\hat{R}(U)|S(T)$$

where stability factor $S(T) = \frac{\int_0^T |\dot{\phi}| dt}{e_T}$

S gives a quantitative measure of the stability of the equation.

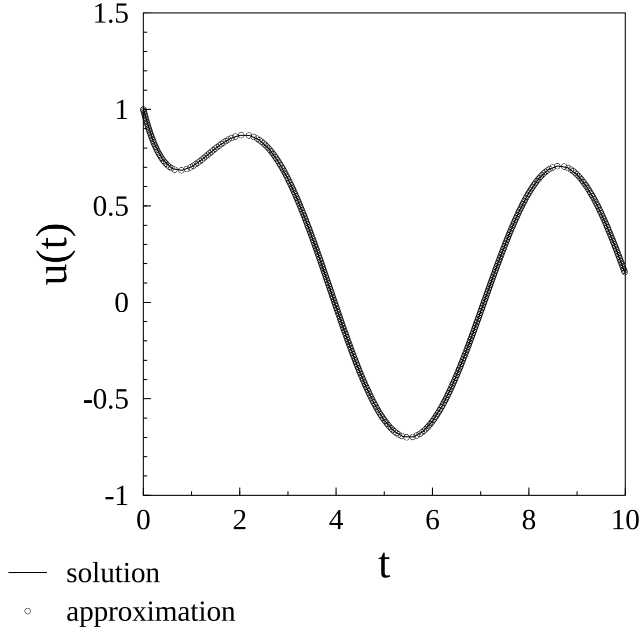
Stability factor examples

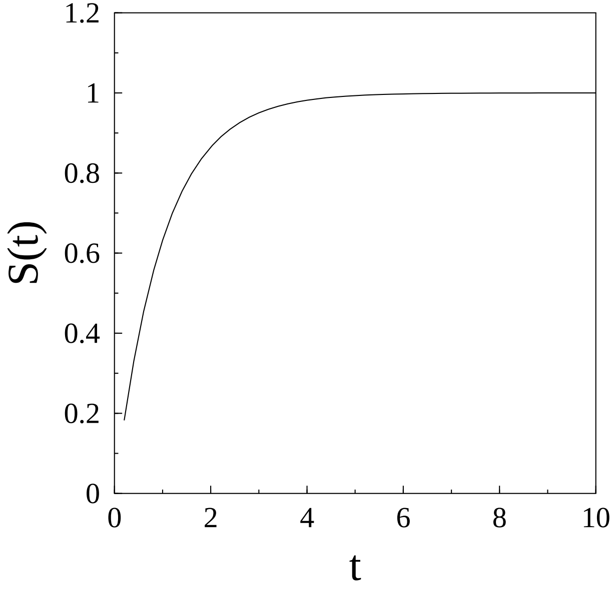
Primal equation

$\dot{u} + u = sin(t), \quad u(0) = u_0$

Dual equation

$-\dot{\phi} + u = 0, \quad \phi(T) = e_T$





S doesn't grow with $T \Rightarrow$ equation is very stable/*parabolic*.

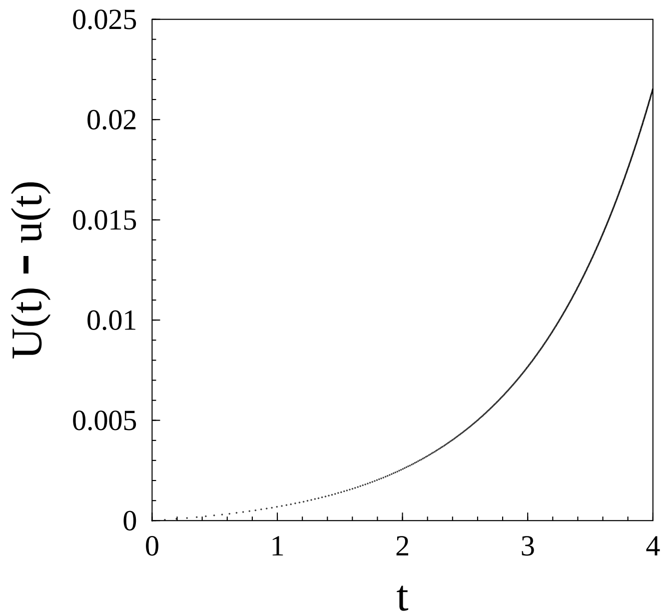
Stability factor examples

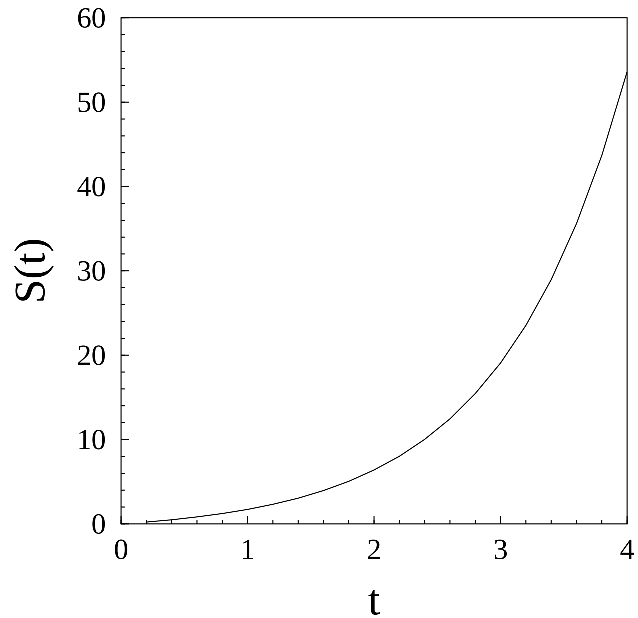
Primal equation

$\dot{u} - u = f(t), \quad u(0) = u_0$

Dual equation

$-\dot{\phi} - u = 0, \quad \phi(T) = e_T$





S grows exponentially with $T \Rightarrow$ very unstable/expensive to compute accurately.

FEM for IBVP/PDE (space-time)

Model problem (Heat equation):
$\dot{u} - \Delta u = f(t, x)$

Domain D is cartesian product of domain in space and time interval: $D = \Omega \times I$

Mesh is space-time slab $D_n = T_n \times I_n$ where $T_n = K$ is triangulation of Ω and I_n is sub-interval of length $k_n = t_n - t_{n-1}$.

cG(1)dG(0): Trial and test space $\bar{W}_k$ with basis functions $\bar{\phi} = \phi_t \phi_x$, where ϕ_t basis functions of W_k (piecewise constant/discontinuous in time) and ϕ_x basis functions of V_h (piecewise linear/continuous in space).

FEM for IBVP/PDE (space-time)

Galerkin's method for cG(1)dG(0):

$$(U_n, v) = (U_{n-1}, v) - \int_{t_{n-1}}^{t_n} (\nabla U, \nabla v)) + (f, v)dt, \quad \forall v \in \bar{W}_k$$

$$U = \sum_{i=1}^{N} \xi_i \bar{\phi} = \sum_{i=1}^{N} \xi_i \phi_{xi}, \quad v = \phi_{xi}, \quad i = 1, ..., N$$

Again, with right-endpoint quadrature we get backward Euler (+quad. err.):

$$(U_n, v) = (U_{n-1}, v) - k_n(\nabla U(t_n, x), \nabla v)) + (f(t_n), v), \quad \forall v \in \bar{W}_k$$

FEM for IBVP/PDE (space-time)

Substituting U gives the formulas for matrix/vector elements.

Stability

Sensitivity to solution (or derivatives of solution) to perturbations in data f and $u0$.
Want to find bounds of type (for exact solution):
$\|u\| \leq S\|f\|$ or $\|u\| \leq S\|u(0)\|$
and for approximate solution:
$\|U\| \leq S\|f\|$ or $\|U\| \leq S\|U(0)\|$
where S is a constant/factor which doesn't depend on u or U.

Heat equation

$$\dot{u} - \Delta u = f(t,x)$$
$$u(0) = u_0$$

Assume $f = 0$

$u(T) + 2\int_0^T \|\nabla u\|^2 dt = \|u(0)\|^2$
Dissipation of temperature/energy (u).

$\|U_n\|^2 + 2k\|\nabla U\|^2 \le \|U_{n-1}\|^2$
Similar statement for discrete solution.
$\|U_n\|^2 \le \|U_{n-1}\|^2$
Discrete solution does not grow for any time step.

Wave equation / cG(1)cG(1)

$$
\begin{aligned}
\ddot{u} - \Delta u &= f(t,x) \\
u(0) = u_0, &\quad \dot{u}(0) = \dot{u}_0 \\
u(x) = 0, &\quad x \in \Gamma
\end{aligned}
$$

or

$$
\begin{aligned}
\dot{u}_1 - \Delta u_2 &= 0 \\
\Delta \dot{u}_2 + \Delta u_1 &= f(t,x) \\
u(0) &= u_0 \\
u(x) &= 0, \quad x \in \Gamma
\end{aligned}
$$

Wave equation / cG(1)cG(1)

Assume $f = 0$

$D_t(\|\dot{u}\|^2 + \|\nabla u\|^2) = 0$
Total energy conserved.

$\|U_n\|\|\| + \|\nabla U_n\| = \|U_{n-1}\|\|\| + \|\nabla U_{n-1}\|$
Also total energy for discrete solution conserved.

Stability factor examples

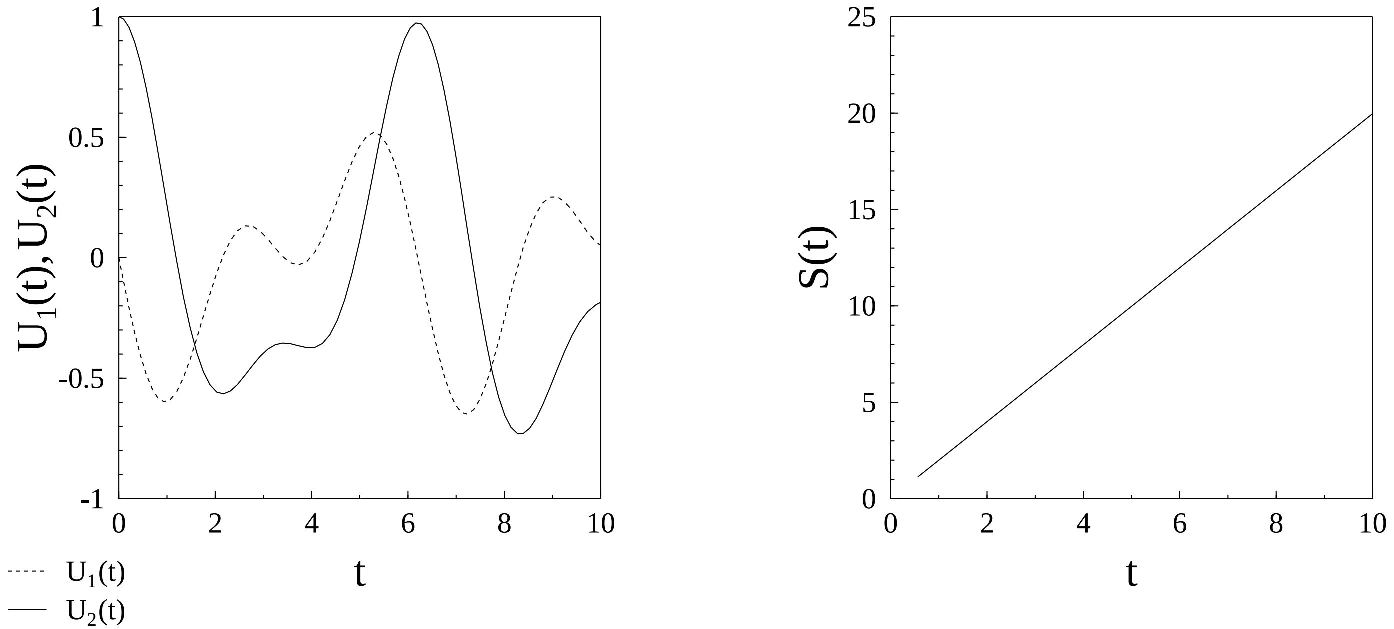
Primal equation

$$
\begin{aligned}
\dot{u}_1 + 2u_2 &= cos(\pi t/3) \\
\dot{u}_2 - 2u_1 &= 0 \\
u_1(0) = 0, \quad u_2(0) &= 1
\end{aligned}
$$

Dual equation

$$
\begin{aligned}
-\dot{\phi}_1 + 2\phi_2 &= 0 \\
-\dot{\phi}_2 - 2\phi_1 &= 0 \\
u_1(0) = \psi_0, \quad u_2(0) &= \psi_1
\end{aligned}
$$

Stability factor examples



S grows linearly with $T \Rightarrow$ hyperbolic.

Implementation of adaptivity