FEM12 - lecture 8

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Contents

Abstract formulation

Framework for linear elliptic PDE.

1. Function space (Hilbert space) V with norm $\|v|_V$ and inner product $(v, w)_V$.
2. Bilinear form $a : V \times V \to R$; determined by underlying differential equation.
3. Linear form $L : V \to R$ determined by data.

Search for solution in function space V, formulate equation using a and L.

Linear form

A linear form $L(v)$ is a function taking values (functions) in V into R: $L(v) \in R \forall v \in V$.
$L(v)$ linear in the argument:

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$$
$$\forall \alpha_i \in R, v_i, \in V$$

Bilinear form

A bilinear form $a(v, w)$ is a function taking values (functions) in $V \times V$ into R: $a(v, w) \in R \forall v, w \in V$.
$a(v, w)$ linear in each argument:

$$
\begin{aligned}
a(\alpha_1 v_1 + \alpha_2 v_2, w_1) &= \alpha_1 a(v_1, w_1) + \alpha_2 a(v_2, w_1) \\
a(v_1, \alpha_1 w_1 + \alpha_2 w_2) &= \alpha_1 a(v_1, w_1) + \alpha_2 a(v_2, w_1) \\
&\forall \alpha_i \in R, v_i, w_i \in V
\end{aligned}
$$

Existence and Uniqueness

Find $u \in V$ such that:
$a(u, v) = L(v) \quad \forall v \in V$
Problem: do such solutions $u \in V$ exist? (Existence)
Problem: if so, is u unique? (Uniqueness)

For certain requirements (a certain class of equations: linear and elliptic) on $a(v, w)$, $L(v)$ and V, we can show existence and uniqueness of solutions u.

Function space: Sobolev space

Natural space for second-order elliptic differential equations:

Bounded function and gradient:
$V = \{v : \int_\Omega (|\nabla v|^2 + v^2) dx < C\}$

Inner product:
$(v, w)_V = \int_\Omega (\nabla v \cdot \nabla w + vw) dx$

Generates the norm:
$\|v\|_V = \sqrt{\int_\Omega (|\nabla v|^2 + v^2) dx}$

Usual notation: $V = H^1(\Omega)$ where index 1 denotes one derivative (H^2 would add a term with the second derivative).
Named after Sergei Sobolev.

Form requirements

Coercivity (or V-ellipticity) of $a(v,w)$:
There exists constant $\kappa_1 > 0 : a(v,v) \geq \kappa_1 \|v\|_V$

Continuity of $a(v,w)$:
There exists constant $\kappa_2 > 0 : |a(v,w)| \leq \kappa_2 \|v\|_V \|w\|_V$

Continuity of $L(v)$:
There exists constant $\kappa_3 > 0 : |L(v)| \leq \kappa_3 \|v\|_V$

Form requirements

Discussion of continuity (by linearity):
$|L(v)| \leq \kappa_3 \|v\|_V \Rightarrow |L(v) - L(w)| \leq \kappa_3 \|v - w\|_V$
Recall Lipschitz continuity for a function $f : R \to R$:
$|f(x) - f(y)| \leq L|x - y|$

In this course we also require that $a(v, w)$ be symmetric when discussing existence/uniqueness.

Lax-Milgram's theorem

If $a(v, w)$ is a continuous, coercive, bilinear form on V and $L(v)$ is a continuous, linear form on V, then there exists unique $u \in V$ satisfying $a(u, v) = L(v), \quad \forall v \in V$.

Poincaré's inequality

$$\|v\|^2_{L_2(\Omega)} \leq C(\|v\|^2_{L_2(\Gamma)} + \|\nabla v\|^2_{L_2(\Omega)})$$

for $v = 0, x \in \Gamma \Rightarrow \|v\|_{L_2(\Gamma)} = 0$:

$$\|v\|_{L_2(\Omega)} \leq C\|\nabla v\|_{L_2(\Omega)}$$

Lax-Milgram's theorem

Poisson example

Abstract Galerkin method

Abstract Galerkin method:

Choose $V_h \in V$ (V_h finite dimensional space)

Find $U \in V_h$:
$a(U, v) = L(v), \quad \forall v \in V_h$

Galerkin orthogonality:
We have $a(u - U, v) = 0, \quad \forall v \in V_h$.