Examination 2D1266, December 20, 2002, 14-19.

Mathematical Models, Analysis and Simulation, Part I, Fall '02

Closed book examination.

Read all the questions first. They are not necessarily sorted in order of ascending difficulty. Work the easy ones first. Good luck!

A sum of credits of 50+/- a few, homework included, is certainly enough to pass. The results will be posted on the bulletin board, Lindstedtsv. 3, floor 3, no later than Jan. 2, 2003. The papers are kept for a year at the NADA expedition and then destroyed. Complaints to J.Oppelstrup/L.Edsberg by Feb. 1, 2003, after which the results are considered irrevocable.

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1 Find an $\mathbf{LDL}^T$-factorization, if possible, of

(3) a)
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 11 \\ 3 & 11 & 25 \end{pmatrix}$$
Is $\mathbf{A}$ definite ?

(2) b) For which a does $\mathbf{Ax} = (1,a,3a)^T$ have at least one solution?

(3) c) Find the solution to $\mathbf{Ax} = (2,4,10)^T$ which has smallest l_2 norm.

2 Consider the conservation law $c_t + Q_x = 0$ for the concentration $0 \le c(x,t) \le 1$ of particles settling under gravity in a container, $Q = c(1 - c)$. At $t = 0$, $c(x,0) = f(x)$.

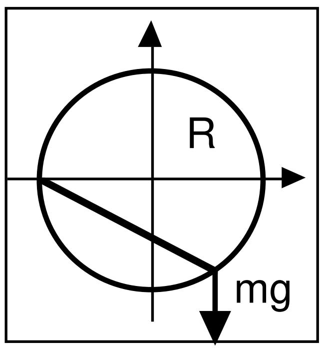
(2) a) Write the equations for the characteristics.

(2) b) Consider $f(x) = x(1-x)$ for $0 \le x \le 1$, 0 elsewhere. Sketch the characteristics in the (x,t)-diagram, starting at $t = 0$, $x = -0.5$, ... , 1.5.

(3) c) Explain with a sketch the formulas for the solution
$$x = z + (1 - 2f(z))\, t$$
$$c(x,t) = f(z)$$
Why do they alone not define the solution uniquely for all (x,t)? Give examples of further conditions used to fix a unique solution.

(2) d) Now let $f(x)$ be a step function, $= c_L$ for $x \le 0$, $= c_R$ for $x > 0$. What is the condition on c_L, c_R that the solution $c(x,t)$ be a shock? What is its speed?

3 The mass m moves without friction on a circular wire, tied to the point $(-R,0)$ by a spring with spring constant K and natural length 0 (zero). Describe the position by Cartesian coordinates (x,y).



(2) a) Write the sum W of the gravitational potential energy and the elastic energy stored in the spring. The equilibrium position can now be found by minimizing W under the constraint that (x,y) be on the wire.

(3) b) Write the Lagrangean for the constrained minimization problem.

(3) c) Perform the minimization over (x,y) and write the corresponding unconstrained dual maximization problem.

(2) d) Find the solution.

4 Consider the differential system with ω positive,

$$\begin{cases} \dfrac{dx}{dt} = x + \omega y - x\sqrt{x^2 + y^2} \\ \dfrac{dy}{dt} = y - \omega x - y\sqrt{x^2 + y^2} \end{cases}$$

(2) a) Prove that the origin is the only critical point.

(3) b) Write the linarization around the origin and find the type of local phase portrait.

(4) c) Write the equations in polar coordinates $x(t) = r(t)\cos\phi(t)$, $y(t) = r(t)\sin\phi(t)$
Use the r-equation to prove that no solution escapes to infinity. Describe the solution in words, and sketch the global phase portrait. What is this kind of limiting behavior called?

5 A horizontal beam of length L, mass per unit length m and uniform flexural rigidity EI is placed on an elastic foundation with stiffness K. It deflects under gravity g. One end has zero displacement ($w(0) = 0$) and deflection angle ($w'(0) = 0$). The equilibrium shape $w(x)$ minimizes (do not prove this)

$$W = \int_0^L \left(\frac{1}{2}EIw''^2 + \frac{1}{2}Kw^2 + mgw\right)dx \text{ subject to } w(0) = 0, w'(0) = 0.$$

(2) a) Explain what the three terms of W represent.

(3) b) Write the Euler-Lagrange differential equation for $w(x)$, disregarding the boundary conditions for the moment. How many conditions are needed?

(4) c) Derive the boundary conditions at $x = L$. A safe way is to consider $W(w+\delta v)$ and write out all the terms in $dW/d\delta = 0$ at $\delta = 0$. You know from $w(0) = w'(0) = 0$ that $v(0) = v'(0) = 0$. There are no restrictions on v, or known conditions for w, at $x = L$. $w''(L)$ and $w'''(L)$ may be present in the conditions.

(2) d) If you did not do c) you may use $w''(L) = 0$, $EIw'''(L) = 0$ in the following:
What is $w(L)$ when $K = 0$?

6 The initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, u(x,0) = f(x) = x(1-x), u(0,t) = 0, u(1,t) = 1.$$

is discretized by the Method of Lines,

$$u_h(x,t) = \sum_{j=1}^{N-1} u_j(t)\phi_j(x) + a\phi_0(x) + b\phi_N(x)$$

with basis functions defined by the standard piecewise linear elements of length h, nodes $x_j = jh$, $j = 0,1,\ldots,N-1,N$, $Nh = 1$. We obtain a linear differential system

$$\mathbf{M}\mathbf{u}_t - \mathbf{A}\mathbf{u} = \mathbf{b},\ \mathbf{u} = (u_1,\ldots,u_{N-1})^T$$

(3) a) What are the values of a and b? What are the initial values $\mathbf{u}(0)$? (Give formulas with the $f(x)$ defined above)

(3) b) Give formulas for the elements of $\mathbf{A}$ and $\mathbf{M}$ in terms of basis functions.

(4) c) Give formulas for the elements of matrix $\mathbf{M}$ and vector $\mathbf{b}$. (i.e., evaluate the integrals appearing in b)).