Chapter 6: Fast Fourier Transform and Applications

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Mathematical Models, Analysis and Simulation, Part I

Fourier Sine Series

Read: Strang, Ch. 4.1

- In the following, every function $f:[-\pi,\pi]\to\mathbb{C}$ will be identified with the periodic continuation onto $\mathbb{R}$.
- A function $f:[-\pi,\pi]\to\mathbb{C}$ is called *odd* if $f(x)=-f(-x)$ for all $x\in[\pi,\pi]$.
- A function f is called *even* if $f(x)=f(-x)$ for all $x\in[\pi,\pi]$.
- If f is even, f' is odd. Similarly, if f is odd, f' is even.
- Most important odd functions: $\sin(nx)$.
- Fourier sine series:

$$\boxed{S(x)=\sum_{n=1}^{\infty} b_n \sin nx}$$

Q: Which functions f can be represented by a sine series?

A: Very many (if “representation” is understood in the correct way).

Computation of the Coefficients

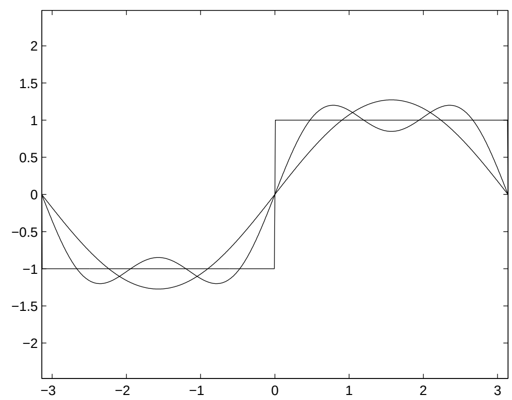
- From $\sin nx \sin kx = 1/2\cos(n-k)x - 1/2\cos(n+k)x$ it follows
$$\int_{-\pi}^{\pi} \sin nx \sin kx dx = \begin{cases} 0, & \text{if } n\neq k,\\ \pi, & \text{if } n=k.\end{cases}$$
The functions $\sin nx$ are orthogonal to each other in $L^2(-\pi,\pi)$.
- Assume that
$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx.$$
Multiply through by $\sin kx$ and integrate:
$$\int_{-\pi}^{\pi} f(x)\sin kx dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin kx dx = b_k \int_{-\pi}^{\pi} \sin^2 kx dx = \pi b_k.$$
So
$$\boxed{b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx.}$$
Assumptions:
  - The series must converge in such a sense that “integration” is possible after multiplication by $\sin kx$.
  - Summation and integration must be exchangable.

A First Example: Square Wave

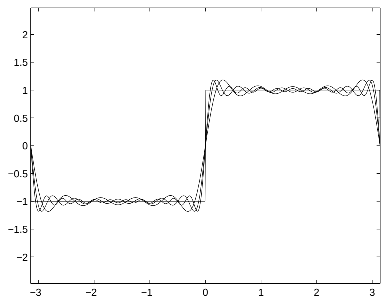
$$SW(x)=\begin{cases} -1, & \text{if } x\in(-\pi,0),\\ 1, & \text{if } x\in(0,\pi),\\ 0, & \text{if } x=-\pi,0,\pi.\end{cases}$$

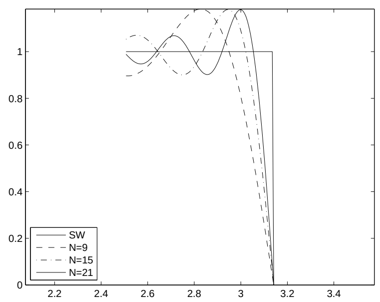
Fourier sine series:

$$SW(x)=\frac{4}{\pi}\left[\frac{\sin x}{\mathbf{1}}+\frac{\sin 3x}{\mathbf{3}}+\frac{\sin 5x}{\mathbf{5}}+\frac{\sin 7x}{\mathbf{7}}+\cdots\right]$$



Gibbs Phenomenon





Gibbs phenomenon: Partial sums overshoot near jumps.

Fourier Cosine Series

- In the case of *even* functions, the prototypes are cosines, $\cos nx$.
- Fourier cosine series:

$$C(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

- Again, we have orthogonality:

$$\int_{-\pi}^{\pi} \cos nx \cos kx dx = \begin{cases} 0, & \text{if } n \neq k, \\ 2\pi, & \text{if } n = k = 0, \\ \pi, & \text{if } n = k > 0. \end{cases}$$

- Let $f : [-\pi, \pi] \to \mathbb{C}$ be an even function. Assume

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

Then:

$$a_k = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, & \text{if } k = 0, \\ \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, & \text{if } k > 0. \end{cases}$$

Examples

Repeating Ramp RR is obtained by integrating SW:

$$RR(x) = |x|.$$

Fourier cosine series:

$$RR(x) = \frac{\pi}{2} - \frac{\pi}{4} \left[\frac{\cos x}{\mathbf{1}^2} + \frac{\cos 3x}{\mathbf{3}^2} + \frac{\cos 5x}{\mathbf{5}^2} + \frac{\cos 7x}{\mathbf{7}^2} + \cdots \right]$$

Note: The coefficients are equal to those obtained by termwise intergration of the sine series for SW.

Up-Down UD is obtained as the derivative of SW:

$$UD(x) = 2\delta(x) - 2\delta(x - \pi).$$

Fourier cosine series:

$$UD(x) = \frac{4}{\pi}[\cos x + \cos 3x + \cos 5x + \cos 7x + \cdots]. \quad \textbf{??}$$

Q: What about convergence? The terms are not a zero sequence!

An Observation: Decay of Coefficients

coefficients	functions
no decay	Delta functions
$1/k$ decay	Step functions (with jumps)
$1/k^2$ decay	Ramp functions (with corners)
$1/k^4$ decay	Spline functions (jumps in f''')
r^k decay ($r < 1$)	Analytic functions

The partial sums for analytical functions converge *exponentially fast*! This is the basis for fast solution methods for certain partial differential equations.

Details will follow later.

Fourier Series For Dirac's Delta Functional

Definition: For every continuous function f on $[-\pi,\pi]$,

$$\int_{-\pi}^{\pi} \delta(x) f(x) dx = f(0).$$

δ is not a usual function. It is a *functional*: $\delta : C[-\pi,\pi] \rightarrow \mathbb{C}$.

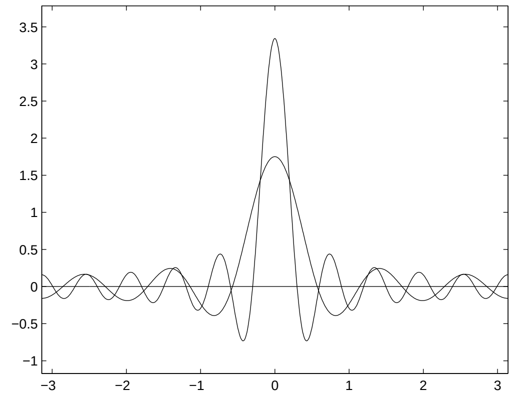
A simple calculation gives:

$$\boxed{\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}[\cos x + \cos 2x + \cos 3x + \cdots].}$$

Partial sums:

$$\delta_N = \frac{1}{2\pi}[1 + 2\cos x + \cdots + 2\cos Nx]$$

Dirac's Delta Functional (cont)



Q: In which sense converges δ_N against δ?

A: For every continuous function $f \in C[-\pi,\pi]$, it holds

$$\int_{-\pi}^{\pi} \delta_N(x) f(x) dx \longrightarrow \int_{-\pi}^{\pi} \delta(x) f(x) dx = f(0).$$

Notation: Weak convergence.

Fourier Series: General Periodic Functions

Let $f : [-\pi,\pi] \rightarrow \mathbb{C}$ be any (nice) function. Then

- $f = f_{\text{even}} + f_{\text{odd}}$
- $f_{\text{even}} = 1/2(f(x) + f(-x))$
- $f_{\text{odd}} = 1/2(f(x) - f(-x))$

Hence, f can be written as a sum of sine and cosine series:

$$\begin{aligned} f &= f_{\text{even}} + f_{\text{odd}} \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \end{aligned}$$

Fouries Series: The Complex Version

- Moivre's theorem: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
- Define $c_k = (a_k - ib_k)/2$, $c_{-k} = (a_k + ib_k)/2$.
- Then:

$$\begin{aligned} c_k e^{ikx} + c_{-k} e^{-ikx} &= c_k(\cos kx + i\sin kx) + c_{-k}(\cos kx - i\sin kx) \\ &= (c_k + c_{-k})\cos kx + i(c_k - c_{-k})\sin kx \\ &= a_k \cos kx + b_k \sin kx. \end{aligned}$$

- The Fourier series can be equivalently written as

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.}$$

In what follows we will always use the complex notation!

Properties

- Let $\phi_k(x) = \exp(ikx)$ for $k = \dots, -2, -1, 0, 1, 2, \dots$.
- Every $f \in L^2(-\pi,\pi)$ (complex!) has a representation

$$f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx},$$

$$\text{with } \hat{f}_k = \frac{1}{\|f\|^2}(f,\phi_k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

  in the sense of $L^2(-\pi,\pi)$. (The partial sums converge towards f in the means square norm.)
- Since $SW \in L^2(-\pi,\pi)$, we conclude that pointwise convergence cannot always be expected.
- Since $e^{ikx} = \cos kx + i\sin kx$, the convergence will be the better the "more periodic" u is.

Further Useful Properties

- Orthogonality in $L^2(-\pi,\pi)$:

$$(\phi_k,\phi_j) = \int_{-\pi}^{\pi} e^{ikx}e^{-ijx}dx = \int_{-\pi}^{\pi} e^{i(k-j)x}dx = 2\pi\delta_{kj}$$

- Parseval's identity:

$$\boxed{\int_{-\pi}^{\pi} |f(x)|^2 dx = \|f\|^2 = 2\pi \sum_{k=-\infty}^{+\infty} |\hat{f}_k|^2}$$

  Consequently,

$$f \in L^2(-\pi,\pi) \Leftrightarrow \sum_{k=-\infty}^{+\infty} |\hat{f}_k|^2 < \infty.$$

- For the derivatives, we have

$$f^{(p)}(x) = \sum_{k=-\infty}^{+\infty} (ik)^p \hat{f}_k e^{ikx}$$

- Let $H^p_{\text{per}} = \{v \in H^p(-\pi,\pi) | v \text{ is } 2\pi\text{-periodic}\}$.

$$\boxed{f \in H^p_{\text{per}} \Leftrightarrow \sum_{k=-\infty}^{+\infty} k^{2p}|\hat{f}_k|^2 < \infty.}$$

  This is the generalization of the decay property for Fourier coefficients. (Strang, p. 321, 327)

The Discrete Fourier Transform (DFT)

Read: Strang, Ch. 4.3

Without loss of generality assume the basic interval to be $[0,2\pi]$.

Let $[0,2\pi]$ be subdivided into N equidistant intervals,

$$h = \frac{2\pi}{N}, \quad x_j = jh.$$

For a periodic function, $f(0) = f(2\pi) = f(x_N)$ such that the trapezoidal rule reads

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx \approx \frac{h}{2\pi}\sum_{j=0}^{N-1} f_j e^{-ikx_j} = \frac{1}{N}\sum_{j=0}^{N-1} f_j\left(e^{-ih}\right)^{jk}$$

$$= \frac{1}{N}\sum_{j=0}^{N-1} f_j \bar{w}^{jk} := c_k$$

Here, we used $\boxed{w = e^{ih}}$

The discrete version of the inverse transformation is,

$$\tilde{f}_j = \sum_{k=0}^{N-1} c_k e^{ikx_j} = \sum_{k=0}^{N-1} c_k w^{kj}$$

Theorem. *One transformation is the inverse of the other,*

$$\boxed{f_j \equiv \tilde{f}_j}$$

The Proof

Proof. Compute:

$$\begin{aligned}\tilde{f}_j &= \sum_{k=0}^{N-1} c_k w^{kj} \\ &= \sum_{k=0}^{N-1} \frac{1}{N}\sum_{l=0}^{N-1} f_l \bar{w}^{lk} w^{kj} \\ &= \frac{1}{N}\sum_{l=0}^{N-1} f_l \sum_{k=0}^{N-1} w^{(j-l)k}\end{aligned}$$

Since

$$\sum_{k=0}^{N-1} w^{(j-l)k} = \begin{cases} \frac{1-w^{(j-l)N}}{1-w} = 0, & \text{if } j \neq l, \\ N, & \text{if } j = l, \end{cases}$$

the result follows. □

Some common notation:

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 1 & \cdot & 1 \\ 1 & w & w^2 & \cdot & w^{N-1} \\ 1 & w^2 & w^4 & \cdot & w^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & w^{N-1} & w^{2(N-1)} & \cdot & w^{(N-1)^2} \end{pmatrix}$$

Then it holds:

$$\mathbf{f} = \mathbf{F}\mathbf{c}, \quad \mathbf{c} = \frac{1}{N}\bar{\mathbf{F}}\mathbf{f}, \quad \mathbf{F}^{-1} = \frac{1}{N}\bar{\mathbf{F}}$$

Note: $\mathbf{F}$ is symmetric, but not Hermitian.

The Fast Discrete Fourier Transform (FFT)

- The naive application of the discrete Fourier transform has complexity $O(N^2)$. *(matrix-vector multiplication)*
- Example: $N = 2^{12}$.
  - The naive approach requires $2^{24} \approx 1.7 \cdot 10^7$ complex multiplications.
  - The FFT requires only $6 \times 2^{12} \approx 2.4 \cdot 10^4$ multiplications.
- The basic idea: Let N be a power of 2 and $M = N/2$.

$$f_j = \sum_{k=0}^{N-1} c_k w^{kj} = \sum_{k \text{ even}} c_k w^{kj} + \sum_{k \text{ odd}} c_k w^{kj}$$
$$= \underbrace{\sum_{k'=0}^{M-1} c_{2k'} (w^2)^{k'j}}_{=:f'_j} + w^j \underbrace{\sum_{k''=0}^{M-1} c_{2k''+1} (w^2)^{k''j}}_{=:f''_j}$$

- *f'_j and f''_j are discrete Fourier transform of half the original size M!*

FFT (cont.)

- This formula can be simplified:
  - For $j = 0, \ldots, M-1$: Take it as it stands.
  - For $j = M, \ldots, N-1$: Let $j' = j - M$. It holds $w^M = -1$ and $w^N = 1$:

$$w^{M+j'} = w^M w^{j'}, \quad (w^2)^{kj} = (w^2)^{kj'}.$$

- This gives the identities

$$\left. \begin{aligned} f_j &= f'_j + w^j f''_j \\ f_{j+M} &= f'_j - w^j f''_j \end{aligned} \right\} j = 0, \ldots, M-1$$

This recursion gives rise to a divide-and-conquer strategy.

Computational complexity: $O(N \log N)$

FFT: Computational Complexity

Assumptions:

- The exponentials w^j are precomputed.
- Let $W(N)$ be the number of complex operations for a FFT of length N.

$$W(2M) = 2W(M) + 4M, W(1) = 0.$$

Denote $w_j = W(2^j)$ and $N = 2^n$:

$$w_0 = 0, w_j = 2w_{j-1} + 2 \cdot 2^j.$$

Multiply the equation by 2^{N-j} and sum up:

$$\sum_{j=1}^{n} 2^{n-j} w_j = 2 \sum_{j=1}^{n} 2^{n-j} \left(w_{j-1} + 2^j \right)$$
$$= 2n2^n + \sum_{j=0}^{n-1} 2^{n-j} w_{j-1}$$

Consequently,

$$\boxed{W(N) = w_n = 2n2^n = 2N \text{ ld } N}$$

Shifted DFT

Using the base interval $[0, 2\pi]$ leads to the standard DFT. What happens if we use $[-\pi, \pi]$ instead? (Let N be even.)

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ikjh}$$
$$= \frac{1}{N} \sum_{j=0}^{N/2-1} f_j e^{-ikjh} + e^{i2\pi} \frac{1}{N} \sum_{j=N/2}^{N-1} f_j e^{-ikjh}$$
$$= \frac{1}{N} \sum_{j=0}^{N/2-1} f_j e^{-ikjh} + \frac{1}{N} \sum_{j=N/2}^{N-1} f_j e^{-ik(j-N)h}$$
$$= \frac{1}{N} \sum_{j=0}^{N/2-1} f_j e^{-ikjh} + \frac{1}{N} \sum_{l=-N/2}^{-1} f_l e^{-iklh}$$
$$= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} f_l e^{-iklh}$$

Similarly, for the inverse DFT it holds,

$$f_l = \sum_{k=-N/2}^{N/2-1} c_k e^{iklh}, \quad l = -N/2, \ldots, N/2-1.$$

Order of coefficients:

DFT: $(c_0, c_1, \ldots, c_{N-1})$
shifted DFT: $(c_{N/2}, c_{N/2+1}, \ldots, c_{N-1}, c_0, \ldots, c_{N/2-1})$

This is what matlab's `fftshift` does.

Fourier Integrals

Read: Strang, p. 367–371

Fourier series are convenient to describe *periodic* functions. Equivalently, f must be defined on a finite interval.

Q: What happens if the function is not periodic?

- Consider $f:\mathbb{R}\to\mathbb{C}$. The *Fourier transform* $\hat{f}=\mathcal{F}(f)$ is given by

$$\boxed{\hat{f}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \quad k\in\mathbb{R}.}$$

  Here, f should be in $L^1(\mathbb{R})$.
- Inverse transformation:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk$$

  Note: Often, you will se a more symmetric version by using a different scaling.
- Theorem of Plancherel: $f\in L^2(\mathbb{R})\Leftrightarrow \hat{f}\in L^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(k)|^2dk.$$

Fourier Integrals: The Key Rules

$$\widehat{df/dx}=ik\hat{f}(k)$$
$$\widehat{\int_{-\infty}^{\cdot} f(x)dx}=\hat{f}(k)/(ik)$$
$$\widehat{f(\cdot-d)}=e^{-ikd}\hat{f}(k)$$
$$\widehat{e^{ic\cdot}f}=\hat{f}(k-c)$$

Examples:

Delta functional

$\hat{\delta}(k)=1$ for all $k\in\mathbb{R}$.

Centred square pulse Let

$$f(x)=\begin{cases}1, & \text{if } -L\le x\le L,\\ 0, & \text{if } |x|>L.\end{cases}$$

Then

$$\hat{f}(k)=2\frac{\sin kL}{k}=2L\,\mathrm{sinc}\,kL,$$

where $\mathrm{sinc}\,t=\sin t/t$ is the *Sinus cardinalis* function.

Sampling

Read: Strang, p. 691–693

By using the Fourier transform $\mathcal{F}$ and its inverse, any function can be reconstructed.

Q: Can a function be reconstructed by using only discrete samples?

Obviously, no. The questions becomes the general interpolation problem which does not have a unique solution.

Often, an interpolation problem gets a unique solution if the class of possible interpolants is restricted.

Q: What is the correct class if we stick to Fourier transforms?

Consider one period of a simple harmonic $f(t)=ae^{i(\omega t+\phi)}$. Obviously, one needs (at least) two samples in $[0,\omega/(2\pi))$ for determining the two parameters.

Use now equidistant sampling with *step size* T.

Definition: Nyquist sampling rate: $T=\pi/\omega$.

For a given sampling rate T, frequencies higher than the *Nyquist frequency* $\omega_N=\pi/T$ cannot be detected. A higher frequency harmonic is mapped to a lower frequency one. This effect is called *aliasing*.

The Sampling Theorem

Using an *a-priori* bound on the Fourier transform $\hat{f}$ of a function f, this function can be reconstructed by discrete sampling.

Theorem: (Shannon-Nyquist) Assume that f is band-limited by W, i.e., $\hat{f}(k)=0$ for all $|k|\ge W$. Let $T=\pi/W$ be the Nyquist rate. Then it holds

$$\boxed{f(x)=\sum_{-\infty}^{\infty} f(nT)\,\mathrm{sinc}\,\pi(x/T-n)}$$

where $\mathrm{sinc}\,t=\sin t/t$ is the *Sinus cardinalis* function.

Note: The sinc function is band-limited:

$$\widehat{\mathrm{sinc}}(k)=\begin{cases}1, & \text{if } -\pi\le k\le\pi,\\ 0, & \text{elsewhere.}\end{cases}$$

Sampling Theorem: Proof

Assume for simplicity $W=\pi$.

By the inverse Fourier transform,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(k)e^{ikx}dk=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat f(k)e^{ikx}dk.$$

Define

$$\tilde f(k)=\begin{cases}\hat f(k), & \text{if } -\pi<x<\pi,\\ \text{periodic continuation}, & \text{if } |k|\ge\pi.\end{cases}$$

$\tilde f$ can be represented as a Fourier series:

$$\tilde f(k)=\sum_{n=-\infty}^{\infty}\hat{\tilde f}_n e^{ink},$$

where

$$\hat{\tilde f}_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(k)e^{-ink}dk=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat f(k)e^{-ink}dk=f(-n).$$

Hence, for $-\pi<x<\pi$,

$$\hat f(k)=\tilde f(k)=\sum_{n=-\infty}^{\infty}f(-n)e^{ink}.$$

Proof (cont)

Therefore,

$$\begin{aligned}f(x)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat f(k)e^{ikx}dk\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}f(-n)e^{ink}\right)e^{ikx}dk\\&=\sum_{n=-\infty}^{\infty}f(-n)\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ik(x+n)}dk\\&=\sum_{n=-\infty}^{\infty}f(-n)\frac{\sin\pi(x+n)}{\pi(x+n)}\\&=\sum_{n=-\infty}^{\infty}f(n)\frac{\sin\pi(x-n)}{\pi(x-n)}\end{aligned}$$

Spectral Interpolation

Read: Strang, p. 448–450

For the DFT (on $[0,2\pi]$) we know

$$f(x_j)=\sum_{k=0}^{N-1}c_k e^{ikx_j},\quad j=0,\dots,N-1.$$

Consider the function Πf,

$$\Pi f(x)=\sum_{k=0}^{N-1}c_k e^{ikx},\quad x\in\mathbb{R}.$$

This is an interpolating trigonometric polynomial, the so-called *spectral interpolant*.

Note: Even for real f, Πf is in general complex (with the exception of the grid points x_j, of course).

Q: How can one obtain a real interpolant for a real-valued function?

Spectral Interpolation (cont.)

1. Replace Πf by the shifted interpolant,

$$\Pi_c f(x)=\sum_{k=-N/2}^{N/2-1}c_k e^{ikx},\quad x\in\mathbb{R}.$$

   Note: $\Pi f\neq\Pi_c f$ with the exception of the grid points.
2. Replace $\Pi_c f$ by its symmetrized variant,

$$Pf(x)={\sum_{k=-N/2}^{N/2}}'' c_k e^{ikx},\quad x\in\mathbb{R}.$$

   Here, ${\sum''}_{k=M}^{N}=\frac{1}{2}c_M+c_{M+1}+\cdots+c_{N-1}+\frac{1}{2}c_N$.

This interpolation can be explicitely written down,

$$Pf(x)=p(x)=\sum_{j=0}^{N-1}f_j\,\mathrm{psinc}(x-jh).$$

psinc is the periodic sinc function,

$$\mathrm{psinc}(x)=\frac{1}{N}{\sum_{k=-N/2}^{N/2}}'' e^{ikx}=\frac{\sin(\pi x/h)}{(2\pi/h)\tan(x/2)}$$

Spectral Methods: Differentiation

Idea: Given a function u at discrete points, interpolate by a suitable smooth function $p(x)$ and set $u'(x_j) \approx p'(x)$.

Examples:

1. Piecewise linear interpolation: $u'(x_j) \approx \frac{u_{j+1} - u_j}{h}$
2. Piecewise quadratic interpolation: $u'(x_j) \approx \frac{u_{j+1} - u_{j-1}}{2h}$

Let's now use spectral interpolation:

$$p(x) = \sum_{j=0}^{N-1} u_j \,\mathrm{psinc}(x - jh)$$

$$u'(x_j) \approx p'(x_j) = \sum_{j=0}^{N-1} u_j \frac{d}{dx} \mathrm{psinc}(x_j - jh)$$

Remarks:

- Piecewise polynomial interpolation uses only *local* informations.
- Spectral differentiation uses all gridpoints for evaluating one derivative.
- Spectral differentiation leads to *full* matrices DP while standard differences give rise to *sparse* matrices.
- Computational complexity:
  - Polynomial: $O(N)$
  - Spectral via FFT: $O(N \log N)$

Repeat: The Finite Element Method

- Start with a differential equation.
- Derive the weak formulation

  $$a(u,v) = L(v) \text{ for all } v \in V \qquad \textbf{(V)}$$

  and, if possible, the minimization formulation

  $$P(u) = \frac{1}{2}a(u,u) - L(u) \to \min! \qquad \textbf{(M)}$$

- Find an approximating (finite dimensional) space $V_h \subset V$ and solve **V** and **M**, respectively: Galerkin and Ritz methods.
- The method yields – up to a constant – the best approximation of u in V_h.

FEM: Pros and Cons

- Advantages with finite elements:
  - Very flexible, easy to adapt to complex domains and/or solutions;
  - Fast algorithms for its implementation, fast solvable by, e.g., multigrid methods;
  - In principle, good convergence: With Pp elements,

    $$\|e_h\| = O(h^{p+1}).$$

- Drawbacks with finite elements:
  - Many degrees of freedom necessary for obtaining a good approximation (especially in 3D);
  - Very hard to construct Pp elements in higher dimensions.

Q: Are there alternatives?

Possible Alternatives

Q: Are there alternatives?

A: Use basis functions which are closely adapted to the problem at hand.

Advantages:

- We will obtain exponential convergence, that is, faster than any power of h.
- Only very few degrees of freedom needed for high accuracy.

Drawbacks:

- The stiffness matrix will be full.
- Every problem needs its own set of ansatz functions.

Q: May it be efficient in practice?

A: Fast transformation algorithms.

An Example

Find the 2π-periodic solutions of

$$-u'' + ru = f(x), x \in (0, 2\pi)$$

with a *constant* $r > 0$.

Weak formulation, with $V = H^1_{\text{per}}(0, 2\pi)$:

$$a(u,v) = \int_0^{2\pi} (u'\bar{v}' + ru\bar{v})dx, \quad L(v) = \int_0^{2\pi} f\bar{v}dx.$$

Insert the Fourier expansion and test against *all* basis functions $\phi_j = e^{ikx}$:

$$\begin{aligned} a(u,\phi_j) &= \int_0^{2\pi} (\sum_{k=-\infty}^{+\infty} ik\hat{u}_k e^{ikx}\overline{ije^{ijx}} + \sum_{k=-\infty}^{+\infty} \hat{u}_k e^{ikx}\overline{e^{ijx}})dx \\ &= \sum_{k=-\infty}^{+\infty} \hat{u}_k \int_0^{2\pi} (ik\cdot(-ij) + r)e^{i(k-j)x}dx \\ &= 2\pi\hat{u}_j(j^2 + r). \end{aligned}$$

Example: The Analytical Solution

$$a(u,\phi_j) = 2\pi\hat{u}_j(j^2 + r).$$

Analogously,

$$L(v) = \int_0^{2\pi} fe^{-ijx}dx = 2\pi\hat{f}_j.$$

Hence,

$$\boxed{\hat{u}_j = \frac{1}{j^2 + r}\hat{f}_j, \quad j = 0, \pm 1, \pm 2, \ldots}$$

- The solution seems even ok if r is not a negative square of an integer.
- If even $f \in H^p_{\text{per}}(0,2\pi)$, then $u \in H^{p+2}_{\text{per}}(0,2\pi)$:

$$\sum_{k=-\infty}^{+\infty} k^{2p+4}|\hat{u}_k|^2 = \sum_{k=-\infty}^{+\infty} k^{2p+4}\frac{|\hat{f}_k|^2}{(k^2+r)^2} < \sum_{k=-\infty}^{+\infty} k^{2p}|\hat{f}_k|^2 < \infty$$

Example: Galerkin's Method Applied

Apply now Galerkin's method with $V_N = \{v | v = \sum_{k=-N/2}^{+N/2} \hat{v}_k e^{ikx}\}$:

Since $\int_0^{2\pi} e^{ikx}e^{ijx}dx = 0$ for $i \neq j$, the solution is easily seen to be:

$$u_N = \sum_{k=-N/2}^{N/2} \hat{u}_{hk}e^{ikx} \text{ with } \hat{u}_{hk} = \hat{u}_k.$$

Error estimation:

$$e_N(x) = u(x) - u_N(x) = \sum_{|k|>N/2} \hat{u}_k e^{ikx}$$

Theorem

- For all square integrable functions f,

$$\boxed{\|e_N\| \leq \frac{16}{N^2}\|f\|}$$

  (quadratic convergence).
- If even $f \in H^p_{\text{per}}(0,2\pi)$:

$$\boxed{O\left(N^{-(p+1)}\right)}$$

- If f is infinitely often differentiable, we have exponential convergence.

Galerkin's Method: Proofs

- For $f \in L^2(0,2\pi)$,

$$\begin{aligned} \|e_N\|^2 &= 2\pi \sum_{|k|>N/2} |\hat{u}_k|^2 = 2\pi \sum_{|k|>N/2} \frac{|\hat{f}_k|^2}{(k^2+r)^2} \\ &\leq \frac{1}{(N^2/4+r)^2}\|f\|^2 \leq \frac{16}{N^4}\|f\|^2 \end{aligned}$$

- If even $f \in H^p_{\text{per}}(0,2\pi)$:

$$\begin{aligned} \|e_N\|^2 &= 2\pi \sum_{|k|>N/2} \frac{|\hat{f}_k|^2}{(k^2+r)^2} = 2\pi \sum_{|k|>N/2} \frac{k^{2p}}{k^{2p}}\frac{|\hat{f}_k|^2}{(k^2+r)^2} \\ &\leq \frac{2\pi}{(N/2)^{2p}(N^2/4+r)^2} \sum_{|k|>N/2} k^{2p}|\hat{f}_k|^2 \leq \frac{C(p)^2}{N^{2p+2}} \end{aligned}$$

What Is Behind It?

Q: Why on earth does this method work that good??

A: The exponentials e^{ikx} are eigenfunctions of the differential operator. (Here: $-u'' + ru$)

Later on, we will see that the discrete versions of the exponentials are eigenfunctions of the finite difference discretizations of certain differential equations.

This makes it clear why they will be important for analyzing numerical schemes.

(Pseudo-)Spectral Methods

Read: Strang, p. 451–453

- Consider again the equation $-u'' + ru = f$.
- Ansatz as before

$$u_N = \sum_{k=-N/2}^{N/2-1} c_k e^{ikx}.$$

- Collocation: Use test functions $v_j(x) = \delta(x - x_j)$ for $x_j = jh - \pi$, $h = 2\pi/N$. Equivalently,

$$-u_N''(x_j) + ru_N(x_j) = f(x_j), \quad j = -N/2, \ldots, N/2-1.$$

- Some computations:

$$\sum_{k=-N/2}^{N/2-1} c_k(-(ik)^2 e^{ikx_j} + re^{ikx_j}) = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j}$$

$$\sum_{k=-N/2}^{N/2-1} [c_k(k^2+r) - \hat{f}_k] e^{ikx_j} = 0, \text{ all } j$$

$$\Longrightarrow \quad c_k(k^2+r) - \hat{f}_k = 0, \quad \text{all } k$$

- The solution becomes

$$c_k = \frac{\hat{f}_k}{k^2+r}.$$

This is the same solution as obtained by the Galerkin method.

Pseudo-Spectral Methods

- Fourier series are only well-suited for periodic boundary conditions.
- In case of Dirichlet boundary conditions, *Chebyshev polynomials* $T_k(x)$ are a viable alternative. (Strang, p. 336–338)
- Chebyshev polynomials are eigenfunctions of the equation

$$-\frac{d}{dx}\left(\frac{1}{w}\frac{dT}{dx}\right) = \lambda wT, \quad -1 < x < 1$$

with $w(x) = \sqrt{1-x^2}$.
- The corresponding scalar product is $(f,g)_w = \int_{-1}^{1} w(x) f(x) g(x) dx$.

Other Applications Of FFT

- Digital signal processing
- Digital image processing (encoding [JPEG, MPEG, DVB], denoising, reconstruction, ...)
- Analysis of random processes
- Stability analysis of numerical schemes