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Graph models and Kirchoff's laws. Modified nodal analysis.

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A system in equilibrium

From earlier: Consider a system of springs and masses in equilibrium.

To obtain our system of equations, we applied three equations (Strang, sec. 2.1):

1) The forces should be in equilibrium. $\mathbf{f} = \mathbf{A}^T\mathbf{w}$
2) Hooke's law for springs. $\mathbf{w} = \mathbf{Ce}$.
3) Relation between elongation of springs and displacements of masses $\mathbf{e} = \mathbf{Au}$.

($\mathbf{u}$ displacements, $\mathbf{w}$ tension in the springs (internal forces), $\mathbf{e}$ elongation of the springs, $\mathbf{f}$ external forces on the masses.)

The system is on the form: $\begin{bmatrix} -\mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$, where $\mathbf{C}$ is a diagonal matrix with positive entries on the diagonal.

This yields $\mathbf{A}^T\mathbf{CAx} = \mathbf{f} + \mathbf{A}^T\mathbf{Cb}$.
"Stiffness matrix" $\mathbf{K} = \mathbf{A}^T\mathbf{CA}$ is symmetric positive definite if the columns of $\mathbf{A}$ are linearly independent.

Incidence matrix A

A is the so called "incidence matrix". [NOTES - EXAMPLES]

In our example, the structure was simply a line. Can have a "graph" instead. Will develop these graph models further and apply it to electric circuits. (Strang, Section 2.4).

Numbers u_i can represent the heights of the nodes, the pressure at the nodes, or the voltages at the nodes, dependent on the applications. Common language: "potentials".
Au: the "potential difference", i.e the difference across an edge.

The nullspace of **A** contains the vector that solve $\mathbf{Au} = 0$. If we do not set any potential u_i, the nullspace is a line. The columns of **A** is a line, it has dimension 1. The columns of **A** are linearly dependent and $\mathbf{A}^T\mathbf{A}$ is singular.

In our example with masses, we fixed one location. For electrical circuits, we say that we "ground one node". This will remove one column of **A**, and the remaining columns will be linearly independent. Then $\mathbf{A}^T\mathbf{A}$ is invertible.

Kirchoff's laws

Consider one node with brach currents $w_1, \dots, w_l$ entering this node.

- Kirchoff's current law (KCL): $w_1 + \cdots + w_l = 0$.

Sum of all branch current entering a node equals zero.

Consider one loop with brach voltages (or voltage drops) $e_1, \dots, e_n$.

- Kirchoff's voltage law (KVL): $e_1 + \cdots + e_n = 0$.

Sum of all branch voltages in a loop equals zero.

In a practical network, many nodes and loops. Need a systematic way to derive all these equations for a given network.

Equations

Kirchoff's current law (KCL) yields

$$\mathbf{A}^T\mathbf{w} = \mathbf{0}.$$

Kirchoff's voltage law is automatically satisified with

$$\mathbf{e} = -\mathbf{A}\mathbf{u}.$$

Ohm's law yields

$$\mathbf{w} = \mathbf{C}\mathbf{e},$$

where $\mathbf{C}$ is a diagonal matrix with entries $c_i > 0$, where c_i is the conductance in branch i.
(have $c_i = 1/R_i$, where R_i is the resistance value. $e_i = R_i w_i$).

Adding current and voltage sources

Adding voltage sources and current sources to the system, yields the modifications:

- $\mathbf{A}^T\mathbf{w} = \mathbf{f}$ (current source in $\mathbf{f}$) ,
- $\mathbf{e} = \mathbf{b} - \mathbf{A}\mathbf{u}$, (voltage source in $\mathbf{b}$).
- Ohm's law, $\mathbf{w} = \mathbf{C}\mathbf{e}$ remains the same.

As a large system, this reads:

$$\begin{bmatrix} \mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix}.$$

where $\mathbf{C}$ is a diagonal matrix with positive entries on the diagonal (the conductances). $\mathbf{A}$ is the incidence matrix.
This yields

$$\mathbf{A}^T\mathbf{C}\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{C}\mathbf{b} - \mathbf{f}.$$

"Stiffness matrix" $\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A}$ is symmetric positive definite if the columns of $\mathbf{A}$ are linearly independent - "ground a node".

The graph Laplacian matrix

The matrix $\mathbf{A}^T\mathbf{A}$ turns out to have a very specific form:

- On the diagonal:

$$(\mathbf{A}^T\mathbf{A})_{jj} = \text{degree} = \text{number of edges meeting at node } j.$$

- Off the diagonal:

$$(\mathbf{A}^T\mathbf{A})_{jk} = \begin{cases} -1 & \text{if nodes } j \text{ and} k \text{ share an edge.} \\ 0 & \text{if no edge goes between nodes } j \text{ and } k. \end{cases}$$

And for $\mathbf{A}^T\mathbf{CA}$:

- On the diagonal changes to $(\mathbf{A}^T\mathbf{CA})_{jj}$ = sum of all conductivities in edges meeting at node j.
- Off the diagonal: the -1 changes to the negative of the conductivity in the branch between node j and k.

Modified Nodal Analysis (MNA)

Will switch to notation more common in engineering context.

- Let $\mathbf{A}_R = -\mathbf{A}^T$, with A incidence matrix of Strang, i.e. opposite sign convention and transposed. We will call $\mathbf{A}_R$ the incidence matrix of the resistive branches.
- Introduce $\mathbf{A}_V$ and $\mathbf{A}_I$ for all voltage source and current source branches.
- Assume N_R resistive branches, N_V voltage source branches, N_I current source branches. With m nodes, $\mathbf{A}_R$ is $m \times N_R$, $\mathbf{A}_V$ is $m \times N_V$, $\mathbf{A}_I$ is $m \times N_I$.
- Let
  $\mathbf{i}_v$: branch currents trough voltage sources $(N_V \times 1)$.
  $\mathbf{I}$: vector of values of all current sources $(N_I \times 1)$.
  $\mathbf{E}$: vector of values of all voltage sources $(N_V \times 1)$.

Also, change notation such that the node potentials, denoted u_i, vector $\mathbf{u}$, in Strang, will in MNA be called e_i and $\mathbf{e}$.
And, change notation such that the $\mathbf{C}$ matrix is denoted $\mathbf{G}$.

Modified Nodal Analysis (MNA), contd

We obtain the system

$$\begin{bmatrix} \mathbf{A}_R\mathbf{G}\mathbf{A}_R^T & \mathbf{A}_V \\ \mathbf{A}_V^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{i}_V \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_I\mathbf{I} \\ \mathbf{E} \end{bmatrix}.$$

- $\mathbf{A}_R$, $\mathbf{A}_V$ and $\mathbf{A}_I$ fincidence matrices for resistive, voltage source and current source branches, respectively.
- $\mathbf{G}$ diagonal matrix with conductances (inverse of resistances).
- $\mathbf{I}$: vector of values of all current sources, $\mathbf{E}$: vector of values of all voltage sources.
- $\mathbf{e}$: vector of node potentials, $\mathbf{i}_v$: branch currents trough voltage sources.

NOTES:

- How to get to this system.
- Example from Strang in this notation.

Modified Nodal Analysis (MNA)

- Handout by Dr. Hanke on MNA can be downloaded from the course webpage.
- Translation table there not updated. Conversion should be:

Strang	*MNA*
u	e
e	ν_R
w	i
C	G
A	$-A_R^T$

RLC circuits and AC analysis

R - resistor, L - inductor, C - capacitor.
Voltage drops V, current I. (Strang's notation).
Characteristic equations:

$$V = RI, \quad I = C\frac{dV}{dt}, \quad V = L\frac{dI}{dt}$$

(if capacitance value C and self-inductance value L are constant).

Assume sinusoidal forcing term of fixed frequency ω.
Typical voltage $V(t) = \hat{V}\cos(\omega t) = \Re(\hat{V}e^{j\omega t})$. Current $I(t) = \Re(\hat{I}e^{j\omega t})$.
Since

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t},$$

we get the simple algebraic law

$$\hat{V} = Z\hat{I},$$

with $Z = 1/R$ for resistors, $Z = j\omega L$ for inductors and $Z = 1/(j\omega C)$ for capacitors.

$j\omega$ analysis - Strang's approach

Same system as before:

$$\begin{bmatrix} \mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix}.$$

But now, the diagonal matrix $\mathbf{C}$ contains the complex impedances Z.

Consider an example where branch 1 has a resistor with resistance R_1, branch 2 a inductor with self-inductance L_2, branch 3 a capacitor with capacitance C_3 and finally, branch 4 a resistor with resistance R_4.
Then:

$$\mathbf{C} = \begin{bmatrix} 1/R_1 & & & \\ & 1/(j\omega L_2) & & \\ & & j\omega C_3 & \\ & & & 1/R_4 \end{bmatrix}.$$

$j\omega$ analysis by MNA

Before, we had incidence matrices $\mathbf{A}_R$, $\mathbf{A}_V$ and $\mathbf{A}_I$, for resistive branches, voltage source branches and current source branches, respectively.
Now, introduce also the incidence matrix $\mathbf{A}_C$ for capacitive branches, and $\mathbf{A}_L$ for inductive branches.

The admittances Y are the inverses of the impedances Z.
Introduce the diagonal matrices $\mathbf{Y}_R$, $\mathbf{Y}_C$ and $\mathbf{Y}_L$ that contain the admittances of the resistive, capacitive and inductive branches, respectively.
(Before, $\mathbf{Y}_R = \mathbf{G}$).

The system can now be written as

$$\begin{bmatrix} \mathbf{A}_R\mathbf{Y}_R\mathbf{A}_R^T + \mathbf{A}_C\mathbf{Y}_C\mathbf{A}_C^T + \mathbf{A}_L\mathbf{Y}_L\mathbf{A}_L^T & \mathbf{A}_V \\ \mathbf{A}_V^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{i}_V \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_I\mathbf{I} \\ \mathbf{E} \end{bmatrix}.$$

(Derivation in NOTES on board).

Transient analysis by MNA

Now, let us do the full transient analysis.
We find the equations (notes on board):

$$\mathbf{A}_C\frac{d}{dt}(\mathbf{C}\mathbf{A}_C^T\mathbf{e}) + \mathbf{A}_R\mathbf{G}\mathbf{A}_R^T\mathbf{e} + \mathbf{A}_L\mathbf{i}_L + \mathbf{A}_V\mathbf{i}_V = -\mathbf{A}_I\mathbf{I}$$

$$\mathbf{L}\frac{d\mathbf{i}_L}{dt} + \frac{d\mathbf{L}}{dt}\mathbf{i}_L - \mathbf{A}_L^T\mathbf{e} = \mathbf{0}$$

And, as before

$$\mathbf{A}_V^T\mathbf{e} = \mathbf{E}$$

where $\mathbf{G}$, $\mathbf{C}$ and $\mathbf{L}$ are the diagonal matrices with conductivities, capacitances and self-inductances. Assuming that $\mathbf{C}$ and $\mathbf{L}$ are constant, we get

$$\begin{bmatrix} \mathbf{A}_C\mathbf{C}\mathbf{A}_C^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} \mathbf{e} \\ \mathbf{i}_L \\ \mathbf{i}_V \end{bmatrix} + \begin{bmatrix} \mathbf{A}_R\mathbf{G}\mathbf{A}_R^T & \mathbf{A}_L & \mathbf{A}_V \\ -\mathbf{A}_L^T & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_V^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{i}_L \\ \mathbf{i}_V \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_I\mathbf{I} \\ \mathbf{0} \\ \mathbf{E} \end{bmatrix}.$$