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OF TECHNOLOGY

Fourier Series and the Fast Fourier Transform

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Fourier series of a periodic function

Section 4.1 in Strang.

- Consider a periodic function, with periodic length 2ℓ , i.e.
 $f(x + 2\ell) = f(x)$.
- The full Fourier series of $f(x)$ on the interval $-\ell < x < \ell$ is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

- Can use orthogonality properties of the sin and cos functions to obtain explicit expressions for the coefficients.



Orthogonality properties

The following holds:

$$\int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx = 0, \quad \text{for all } n, m,$$
$$\int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} dx = \begin{cases} 0, & \text{if } n \neq m, \\ \ell, & \text{if } n = m > 0, \\ 2\ell, & \text{if } n = m = 0. \end{cases}$$
$$\int_{-\ell}^{\ell} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx = \begin{cases} 0, & \text{if } n \neq m, \\ \ell, & \text{if } n = m > 0. \end{cases}$$

Verify these properties! Useful trigonometric identities:

$$\begin{aligned} \sin nx \sin kx &= (\cos(n-k)x - \cos(n+k)x)/2, \\ \cos nx \cos kx &= (\cos(n-k)x + \cos(n+k)x)/2, \\ \sin nx \cos kx &= (\sin(n-k)x + \sin(n+k)x)/2, \\ \sin^2 \alpha &= (1 - \cos(2\alpha))/2, \quad \cos^2 \alpha = (1 + \cos(2\alpha))/2. \end{aligned}$$

The coefficients

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

Multiply through by $\cos \frac{k\pi x}{\ell}$ and integrate. Use orthogonality properties. (Different for $k = 0$ and $k > 0$). Yields expression for coefficients

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos kx \, dx, \quad k = 0, 1, 2, \dots$$

Multiply through by $\sin \frac{k\pi x}{\ell}$ and integrate. This yields:

$$b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin kx \, dx, \quad k = 1, 2, \dots$$

Fouries Series: The Complex Version

- Moivre's theorem: $e^{i\alpha} = \cos \alpha + i \sin \alpha$.
- Define $c_k = (a_k - ib_k)/2$, $c_{-k} = (a_k + ib_k)/2$.
- Then:
$$c_k e^{ikx} + c_{-k} e^{-ikx} = c_k (\cos kx + i \sin kx) + c_{-k} (\cos kx - i \sin kx)$$
$$= (c_k + c_{-k}) \cos kx + i(c_k - c_{-k}) \sin kx$$
$$= a_k \cos kx + b_k \sin kx.$$
- The Fourier series can be equivalently written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Odd and even functions

- The function f is odd if $f(x) = -f(-x)$.
- The function f is even if $f(x) = f(-x)$.
- If f is even, then f' is odd. Similarly, if f is odd, then f' is even.

If f is odd, then

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos kx \, dx = 0 \quad \forall k.$$

and $f(x)$ is represented by a Fourier sine series on $(-\ell, \ell)$.
Similarly, if f is even

$$b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin kx \, dx = 0 \quad \forall k.$$

and $f(x)$ is represented by a Fourier cosine series on $(-\ell, \ell)$.

Diffusion equation - separation of variables.

Consider the diffusion equation: $u_t = ku_{xx}$ with initial conditions $u(x, 0) = g(x)$.

Assuming u on the form $u(x, t) = T(t)X(x)$ yields

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{const}$$

The solutions to

$$X'' = -\lambda X, \quad \text{in } 0 < x < 1$$

have the form,

$$X(x) = C \cos \sqrt{\lambda}x + D \sin \sqrt{\lambda}x,$$

and the solution to $T' = -\lambda T$ is $T(t) = e^{-\lambda kt}$.



Diffusion equation - separation of variables, contd

$$X(x) = C \cos \sqrt{\lambda}x + D \sin \sqrt{\lambda}x,$$

With homogeneous Dirichlet boundary conditions, $u(0, t) = u(1, t) = 0$ and hence $X(0) = X(1) = 0$,

$$\lambda_n = (n\pi)^2, \quad X_n(x) = b_n \sin n\pi x, \quad (n = 1, 2, 3, \dots).$$

With homogeneous Neumann boundary conditions, $u_x(0, t) = u_x(1, t) = 0$ and hence $X'(0) = X'(1) = 0$,

$$\lambda_n = (n\pi)^2, \quad X_n(x) = a_n \cos n\pi x, \quad (n = 0, 1, 2, 3, \dots).$$

With periodic boundary conditions, $u(0, t) = u(1, t)$, $u_x(0, t) = u_x(1, t)$ and hence $X(0) = X(1)$, $X'(0) = X'(1)$,

$$\lambda_n = (2n\pi)^2, \quad X_n(x) = a_n \cos 2n\pi x + b_n \sin 2n\pi x, \quad (n = 0, 1, 2, 3, \dots).$$

with $b_0 = 0$.



Symmetric boundary conditions

Consider the eigenvalue problem

$$X'' + \lambda X = 0 \quad \text{in } (a, b)$$

Any pair of boundary conditions

$$\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0$$

$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0$$

are called *symmetric* if

$$f'(x)g(x) - f(x)g'(x) \Big|_{x=a}^{x=b} = 0,$$

for any pair of functions $f(x)$ and $g(x)$ both of which satisfy the pair of boundary conditions.

The Dirichlet, Neumann and periodic BC considered on the previous slide are the most common examples of symmetric boundary conditions.



Orthogonality and General Fourier series

Consider the eigenvalue problem

$$X'' + \lambda X = 0 \quad \text{in } (a, b) \text{ with any symmetric BC}$$

Then, all eigenvalues are real, and eigenfunctions corresponding to different eigenvalues are orthogonal. If λ_n is an eigenvalue with multiplicity k , its k linearly independent eigenfunctions can be rechosen to be orthogonal and real.

I.e. if X_1 and X_2 are any two eigenfunctions ($X_1 \neq X_2$), then

$$(X_1, X_2) = \int_a^b X_1(x)X_2(x)dx = 0.$$

For any function $f(x)$ on (a, b) its Fourier series is

$$\sum_n A_n X_n(x),$$

where the Fourier coefficients are defined as

$$A_n = \frac{(f, X_n)}{(X_n, X_n)}.$$



Notions of convergence

Definition. We say that an infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ *pointwise* in (a, b) if it converges to $f(x)$ for each $a < x < b$. That is, for each $a < x < b$ we have

$$\left| f(x) - \sum_{n=1}^{\infty} f_n(x) \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

Definition. We say that the series *converges uniformly* to $f(x)$ in $[a, b]$ if

$$\max_{a \leq x \leq b} \left| f(x) - \sum_{n=1}^{\infty} f_n(x) \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

Definition. We say that the series *converges in the mean-square (or L^2) sense* to $f(x)$ in (a, b) if

$$\int_a^b \left| f(x) - \sum_{n=1}^{\infty} f_n(x) \right|^2 dx \rightarrow 0 \quad \text{as } N \rightarrow \infty$$



Convergence theorems

Theorem: Uniform convergence.

The Fourier series $\sum_n A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

- i) $f(x)$, $f'(x)$ and $f''(x)$ exist and are continuous for $a \leq x \leq b$, and
- ii) $f(x)$ satisfies the given boundary conditions.

For the classical Fourier series (sine, cosine or full), it is not required that $f''(x)$ exist.

Theorem: Pointwise convergence.

The classical Fourier series (sine, cosine or full) converges to $f(x)$ pointwise on (a, b) , provided that $f(x)$ is a continuous function on $a \leq x \leq b$ and $f'(x)$ is piecewise continuous on $a \leq x \leq b$.

Theorem: L^2 convergence.

The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 dx \quad \text{is finite.}$$



Piecewise continuous functions

- A function $f(x)$ has a jump discontinuity at a point $x = c$ if the one-sided limits $f(c+)$ and $f(c-)$ exist but are not equal. (It does not matter whether $f(c)$ is defined or not).
- A function $f(x)$ is called piecewise continuous on an interval $[a, b]$ if it is continuous at all but a finite number of points and has jump discontinuities at those points.

Theorem: Pointwise convergence.

If $f(x)$ itself is only piecewise continuous on $a \leq x \leq b$ and $f'(x)$ is also piecewise continuous on $a \leq x \leq b$, then the classical Fourier series (sine, cosine or full) converges at every point x ($-\infty < x < \infty$). The sum is

$$\sum_n A_n X_n(x) = \frac{1}{2} [f(x+) + f(x-)] \quad \text{for all } a < x < b.$$

Examples

- Square wave:

$$SW(x) = \begin{cases} -1, & \text{if } x \in (-\pi, 0), \\ 1, & \text{if } x \in (0, \pi), \\ 0, & \text{if } x = -\pi, 0, \pi. \end{cases}$$

Fourier sine series:

$$SW(x) = \frac{4}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \dots \right]$$

Partial sums overshoot near jumps: **Gibbs phenomenon**.

- **Repeating Ramp:** RR is obtained by integrating SW :

$$RR(x) = |x|.$$

Fourier cosine series:

$$RR(x) = \frac{\pi}{2} - \frac{\pi}{4} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \dots \right]$$

Note: The coefficients are equal to those obtained by termwise integration of the sine series for SW .

[ON LAPTOP]

The Gibbs phenomenon

- The Gibbs phenomenon is the overshoot that occurs when we approximate a jump discontinuity - a step function - with a truncated Fourier series. We denote this by a partial sum, S_N , including N terms.
- The oscillations move closer and closer to the jump as we include more terms in the partial sum (N increases), but they do not disappear.
- Gibbs showed that the amplitude of the overshoot approaches a value of 9% of the size of the jump discontinuity.
- The width of the overshoot goes to 0 as $N \rightarrow \infty$, while the extra height remains at 9% (top and bottom).

Again, the complex Fourier series

- Let $\phi_k(x) = \exp(ikx)$ for $k = \dots, -2, -1, 0, 1, 2, \dots$
- Every $f \in L^2(-\pi, \pi)$ (complex!) has a representation

$$f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx},$$

$$\text{with } \hat{f}_k = \frac{1}{\|f\|^2}(f, \phi_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

in the sense of $L^2(-\pi, \pi)$.

- Orthogonality in $L^2(-\pi, \pi)$:

$$(\phi_k, \phi_j) = \int_{-\pi}^{\pi} e^{ikx} e^{-ijx} dx = \int_{-\pi}^{\pi} e^{i(k-j)x} dx = 2\pi \delta_{kj}$$

- The regularity of f determines what type of convergence we can expect.
- "Regularity" also includes regularity across periodic boundary.

Decay of Coefficients

- Parseval's identity:

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \|f\|^2 = 2\pi \sum_{k=-\infty}^{+\infty} |\hat{f}_k|^2$$

Consequently, $f \in L^2(-\pi, \pi) \Leftrightarrow \sum_{k=-\infty}^{+\infty} |\hat{f}_k|^2 < \infty$.

- For the derivatives, we have

$$f^{(p)}(x) = \sum_{k=-\infty}^{+\infty} (ik)^p \hat{f}_k e^{ikx}.$$

- Let $H_{\text{per}}^p = \{v \in H^p(-\pi, \pi) | v \text{ is } 2\pi\text{-periodic}\}$.
(f and the p first derivatives of f are square integrable).

$$f \in H_{\text{per}}^p \Leftrightarrow \sum_{k=-\infty}^{+\infty} k^{2p} |\hat{f}_k|^2 < \infty.$$

This is the generalization of the decay property for Fourier coefficients. (Strang, p. 321, 327)



Decay of Coefficients, contd.

coefficients	functions
no decay	Delta functions
$1/k$ decay	Step functions (with jumps)
$1/k^2$ decay	Ramp functions (with corners)
$1/k^4$ decay	Spline functions (jumps in f''')
r^k decay ($r < 1$)	Analytic functions

I.e, for each additional continuous derivative of f , the power of decay of Fourier coefficients increases by one.

The partial sums for analytical functions (c^∞) converge *exponentially fast*! This is the basis for fast solution methods for certain partial differential equations.



The Discrete Fourier Transform (DFT)

Read: Strang, Ch. 4.3

Without loss of generality assume the basic interval to be $[0, 2\pi]$.

Let $[0, 2\pi]$ be subdivided into N equidistant intervals,

$$h = \frac{2\pi}{N}, \quad x_j = jh.$$

For a periodic function, $f(0) = f(2\pi) = f(x_N)$ such that the trapezoidal rule reads

$$\frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \approx \frac{h}{2\pi} \sum_{j=0}^{N-1} f_j e^{-ikx_j} = \frac{1}{N} \sum_{j=0}^{N-1} f_j (e^{-ih})^{jk}$$
$$[\text{let } w = e^{ih} \text{ and so } \bar{w} = e^{-ih}] = \frac{1}{N} \sum_{j=0}^{N-1} f_j \bar{w}^{jk} := c_k$$

The discrete version of the inverse transformation is,

$$\tilde{f}_j = \sum_{k=0}^{N-1} c_k e^{ikx_j} = \sum_{k=0}^{N-1} c_k w^{kj}$$

DFT, contd.

Theorem. One transformation is the inverse of the other,

$f_j \equiv \tilde{f}_j$

Proof Compute:

$$\tilde{f}_j = \sum_{k=0}^{N-1} c_k w^{kj} = \sum_{k=0}^{N-1} \frac{1}{N} \sum_{l=0}^{N-1} f_l \bar{w}^{lk} w^{kj} = \frac{1}{N} \sum_{l=0}^{N-1} f_l \sum_{k=0}^{N-1} w^{(j-l)k}$$

Since

$$\sum_{k=0}^{N-1} w^{(j-l)k} = \begin{cases} \frac{1-w^{(j-l)N}}{1-w} = 0, & \text{if } j \neq l, \\ N, & \text{if } j = l, \end{cases}$$

the result follows.

The Fast Discrete Fourier Transform (FFT)

- The naive application of the discrete Fourier transform has complexity $O(N^2)$. (*matrix-vector multiplication*)
- Example: $N = 2^{12}$.
 - The naive approach requires $2^{24} \approx 1.7 \cdot 10^7$ complex multiplications.
 - The FFT requires only $6 \times 2^{12} \approx 2.4 \cdot 10^4$ multiplications.
- The basic idea: Let N be a power of 2 and $M = N/2$.

$$\begin{aligned} f_j &= \sum_{k=0}^{N-1} c_k w^{kj} = \sum_{k \text{ even}} c_k w^{kj} + \sum_{k \text{ odd}} c_k w^{kj} \\ &= \underbrace{\sum_{k'=0}^{M-1} c_{2k'} (w^2)^{k'j}}_{=: f'_j} + w^j \underbrace{\sum_{k''=0}^{M-1} c_{2k''+1} (w^2)^{k''j}}_{=: f''_j} \end{aligned}$$

- f'_j and f''_j are discrete Fourier transform of half the original size M !

FFT (cont.)

- This formula can be simplified:
 - For $j = 0, \dots, M-1$: Take it as it stands.
 - For $j = M, \dots, N-1$: Let $j' = j - M$.
 It holds $w^M = e^{i\pi} = -1$ and $w^N = w^{2M} = 1$;

$$w^j = w^{M+j'} = w^M w^{j'} = -w^{j'}, \quad (w^2)^{kj} = (w^2)^{kj'}.$$
- This gives the identities

$$\left. \begin{aligned} f_j &= f'_j + w^j f''_j \\ f_{j+M} &= f'_j - w^j f''_j \end{aligned} \right\} j = 0, \dots, M-1$$

This can be applied recursively: compute each of f'_j and f''_j by applying this formula again. The number of terms in the sums is halved in each step.

With $N = 2^n$, $\log_2(N)$ levels until at "the bottom".

Computational complexity: $O(N \log N)$

FFT: Computational Complexity

Assumptions:

- The exponentials w^j are precomputed.
- Let $W(N)$ be the number of complex operations for a FFT of length N .

$$W(2M) = 2W(M) + 4M, W(1) = 0.$$

Denote $w_j = W(2^j)$ and $N = 2^n$:

$$w_0 = 0, w_j = 2w_{j-1} + 2 \cdot 2^j.$$

Multiply the equation by 2^{N-j} and sum up:

$$\sum_{j=1}^n 2^{n-j} w_j = 2 \sum_{j=1}^n 2^{n-j} (w_{j-1} + 2^j) = 2n2^n + \sum_{j=0}^{n-1} 2^{n-j} w_{j-1}$$

Consequently,

$$W(N) = w_n = 2n2^n = 2N \log_2 N$$

Shifted DFT

Using the base interval $[0, 2\pi]$ leads to the standard DFT. What happens if we use $[-\pi, \pi]$ instead? (Let N be even.)

$$\begin{aligned} c_k &= \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ikjh} = \frac{1}{N} \sum_{j=0}^{N/2-1} f_j e^{-ikjh} + e^{i2\pi} \frac{1}{N} \sum_{j=N/2}^{N-1} f_j e^{-ikjh} \\ &= \frac{1}{N} \sum_{j=0}^{N/2-1} f_j e^{-ikjh} + \frac{1}{N} \sum_{j=N/2}^{N-1} f_j e^{-ik(j-N)h} \\ &= \frac{1}{N} \sum_{j=0}^{N/2-1} f_j e^{-ikjh} + \frac{1}{N} \sum_{l=-N/2}^{-1} f_l e^{-iklh} = \frac{1}{N} \sum_{l=-N/2}^{N/2-1} f_l e^{-iklh} \end{aligned}$$

Similarly, for the inverse DFT it holds,

$$f_l = \sum_{k=-N/2}^{N/2-1} c_k e^{iklh}, \quad l = -N/2, \dots, N/2 - 1.$$

Order of coefficients:

DFT: $(c_0, c_1, \dots, c_{N-1})$

shifted DFT: $(c_{N/2}, c_{N/2+1}, \dots, c_{N-1}, c_0, \dots, c_{N/2-1})$

This is what matlab's `fftshift` does.