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Fourier integrals and the sampling theorem

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Fourier Integrals

Read: Strang, Section 4.5, p. 367–373
Fourier series are convenient to describe *periodic* functions (or functions with support on a finite interval).
What to do if the function is not periodic?

- Consider $f : \mathbb{R} \to \mathbb{C}$. The *Fourier transform* $\hat{f} = \mathcal{F}(f)$ is given by

$$\boxed{\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad k \in \mathbb{R}.}$$

  Here, f should be in $L^1(\mathbb{R})$.
- Inverse transformation:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

  Note: Often, you will se a more symmetric version by using a different scaling.
- Theorem of Plancherel: $f \in L^2(\mathbb{R}) \Leftrightarrow \hat{f} \in L^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk.$$

Fourier Integrals: The Key Rules

$$\widehat{df/dx} = ik\hat{f}(k) \qquad \widehat{\int_{-\infty}^{x} f(\xi)d\xi} = \hat{f}(k)/(ik)$$

$$\widehat{f(x-d)} = e^{-ikd}\hat{f}(k) \qquad \widehat{e^{icx}f} = \hat{f}(k-c)$$

Examples:

Delta functional

$$\hat{\delta}(k) = 1 \text{ for all } k \in \mathbb{R}.$$

Centred square pulse Let

$$f(x) = \begin{cases} 1, & \text{if } -L \le x \le L, \\ 0, & \text{if } |x| > L. \end{cases}$$

Then

$$\hat{f}(k) = 2\frac{\sin kL}{k} = 2L\mathrm{sinc}kL,$$

where $\mathrm{sinc}t = \sin t/t$ is the *Sinus cardinalis* function.

Sampling of functions

Read: Strang, p. 691–693

- By using the Fourier transform $\mathcal{F}$ and its inverse, any function can be reconstructed.
- Assume we know only some discrete values of a function, at a number of discrete points. Can the function be reconstructed?
- Without any assumptions on the function, it is not possible to reconstruct the function.
- Let us consider how many samples we need to identify a harmonic function.

Sampling of functions - The Nyquist sampling rate

Consider a harmonic function such as $\sin \omega t$ or $\cos \omega t$.
Assume equidistant sampling with *step size* T.
The Nyquist sampling rate, is exactly 2 equidistant samples over a full period of the function.
The period is of length $2\pi/\omega$, with 2 samples, yields

Definition: Nyquist sampling rate: $T = \pi/\omega$.

For a given sampling rate T, frequencies higher than the *Nyquist frequency* $\omega_N = \pi/T$ cannot be detected. A higher frequency harmonic is mapped to a lower frequency one. This effect is called *aliasing*.

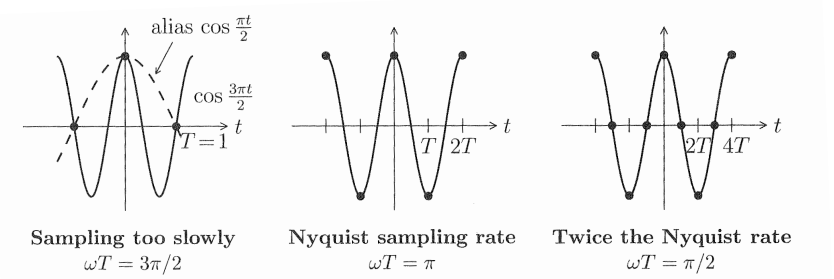

Figure B1: For $\omega T < \pi$, the discrete samples are enough to determine the signal.

Figure from Strang.

The Sampling Theorem

Theorem: (Shannon-Nyquist) Assume that f is band-limited by W, i.e., $\hat{f}(k) = 0$ for all $|k| \geq W$. Let $T = \pi/W$ be the Nyquist rate. Then it holds

$$f(x) = \sum_{-\infty}^{\infty} f(nT)\text{sinc}\pi(x/T - n)$$

where $\text{sinc}t = \sin t/t$ is the *Sinus cardinalis* function.
Note: The sinc function is band-limited:

$$\widehat{\text{sinc}}(k) = \begin{cases} 1, & \text{if } -\pi \leq k \leq \pi, \\ 0, & \text{elsewhere.} \end{cases}$$

The Sampling Theorem: Proof

Assume for simplicity $W = \pi$.
By the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{ikx}dk.$$

Define

$$\tilde{f}(k) = \begin{cases} \hat{f}(k), & \text{if } -\pi < k < \pi, \\ \text{periodic continuation}, & \text{if } |k| \geq \pi. \end{cases}$$

$\tilde{f}$ can be represented as a Fourier series:

$$\tilde{f}(k) = \sum_{n=-\infty}^{\infty} \hat{\tilde{f}}_n e^{ink},$$

where

$$\hat{\tilde{f}}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(k)e^{-ink}dk = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{-ink}dk = f(-n).$$

Hence, for $-\pi < x < \pi$,

$$\hat{f}(k) = \tilde{f}(k) = \sum_{n=-\infty}^{\infty} f(-n)e^{ink}.$$

The Sampling Theorem: Proof, contd.

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)e^{ikx}dk \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} f(-n)e^{ink}\right) e^{ikx}dk \\ &= \sum_{n=-\infty}^{\infty} f(-n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik(x+n)}dk \\ &= \sum_{n=-\infty}^{\infty} f(-n)\frac{\sin \pi(x+n)}{\pi(x+n)} \\ &= \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)} \end{aligned}$$