ROYAL INSTITUTE OF TECHNOLOGY

ODE, part 2

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Dynamical systems, differential equations

Consider a system of n first order equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u}(0) = \mathbf{u}_0,$$

$\mathbf{u} = (u_1(t), \ldots, u_n(t))^T$, $\mathbf{f} = (f_1(\mathbf{u}, t), \ldots, f_n(\mathbf{u}, t))^T$.

- $\mathbf{u}$ is the state-vector.
- Higher order equations can be rewritten as a system of first order equations by introducing new variables for the derivatives.
- Often we write the system in *autonomous* form - i.e with no explicit dependence on t. (i.e. with $\mathbf{f}(\mathbf{u}(t))$ instead of $\mathbf{f}(\mathbf{u}(t), t)$). This is no restriction: Can always add the equation $du_{n+1}/dt = 1$ and use u_{n+1} for t.

Existence and uniqueness

Definition: $\mathbf{f}$ is Lipschitz continuous in D if, for $\mathbf{x}$ and $\mathbf{y}$ in D, there is a constant L such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|.$$

Theorem: Let $\mathbf{f}$ be Lipschitz continuous in an open domain D containing $\mathbf{u}_0$. Then, the initial value problem has a unique, continuously differentiable solution on $[0, T)$ for some $T > 0$.

However, it might not be possible to find the solution and give an analytic expression for it. Then, we can turn to a "Qualitative theory" to learn about properties of the solutions. Do they grow, decay or oscillate? What is the long term behavior of solutions?

Critical points

If $\mathbf{f}(\mathbf{u}^*) = 0$, then we call $\mathbf{u}^*$ a critical point.
Any critical point is a constant solution to the differential equation ($d\mathbf{u}^*/dt = 0$).

One can study the behavior of solutions in a neighbourhood of $\mathbf{u}^*$ by linearization.
Assume that $\mathbf{u}^* + \mathbf{v}(t)$ is a solution, then

$$\frac{d}{dt}(\mathbf{u}^* + \mathbf{v}(t)) = \frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{u}^* + \mathbf{v}) = \underbrace{\mathbf{f}(\mathbf{u}^*)}_{=0} + \mathbf{J}(\mathbf{u}^*)\mathbf{v} + O(\|\mathbf{v}\|^2),$$

with the Jacobian $J_{ij} = \frac{\partial f_i(\mathbf{u})}{\partial u_j}$ evaluated at $\mathbf{u}^*$.

$\mathbf{J}(\mathbf{u}^*)$ is a constant matrix, let us denote it by $\mathbf{A}$.
The dynamics close to the critical point is determined by $\mathbf{A}$, and we can learn a lot from studying the behavior of solutions of the linear system.

Linear constant coefficient 2D systems

Consider

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}.$$

Eigenvalues of $\mathbf{A}$, λ_1 and λ_2. Corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
For diagonalizable systems:

$$\mathbf{u}(t) = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_1 e^{\lambda_2 t},$$

where C_1 and C_2 are determined by the initial condition.
The characteristic equation is

$$0 = det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = \lambda^2 - p\lambda + q,$$

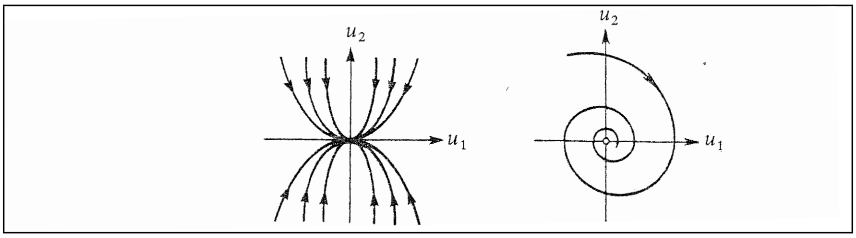
where $p = tr(\mathbf{A})$ and $q = det(\mathbf{A})$.
Form quantity $D = p^2/4 - q$.

- $D > 0$: Real and distinct eigenvalues.
- $D < 0$: Complex conjugate pair of eigenvalues.
- $D = 0$: Double real eigenvalues.

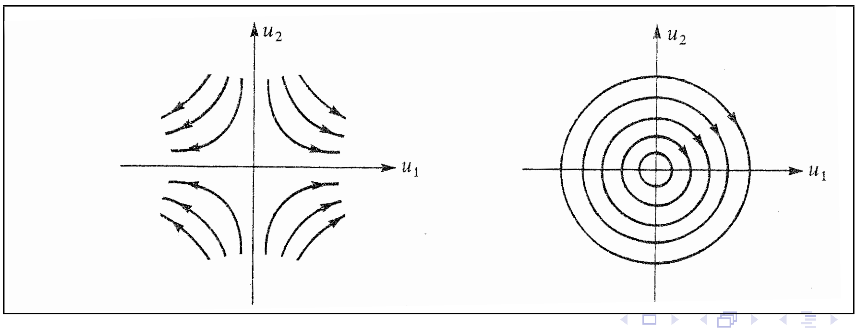
Phase portraits

The solution to the system depends on the initial condition. Plot solution trajectories, starting at several initial values. Does not give a time-scale, t is not in the plots. We plot the solution trajectories, but not how fast they are traveled.

Nodes and spirals



Saddle points and neutrally stable centers



The cases as characterized by D

We had introduced $p = tr(\mathbf{A})$ and $q = det(\mathbf{A})$ and $D = p^2/4 - q$.
The p, q characterization determines all the cases, except the two cases when $D = 0$: diagonalizable and defective (=non-diagonalizable).

1. $D > 0$: Real and distinct eigenvalues. $\mathbf{u}_1 = \mathbf{v}_1 e^{\lambda_1 t}$, $\mathbf{u}_2 = \mathbf{v}_2 e^{\lambda_2 t}$.
   - *i*) Negative. $\lambda_1 < \lambda_2 < 0$.
     Both $\mathbf{u}_1$ and $\mathbf{u}_2$ go to zero as $t \to \infty$. $\mathbf{u}_1$ decays faster than $\mathbf{u}_2$.
     Final approach to 0 along $\mathbf{v}_2$.
     A stable *node*.
   - *ii*) Both signs. $\lambda_1 < 0 < \lambda_2$.
     $\mathbf{u}_1 \to 0$, $\mathbf{u}_2 \to \infty$.
     Final divergence to ∞ along $\mathbf{v}_2$ with $\mathbf{u}_1 = 0$.
     A *saddle point*, unstable, except for points with $\mathbf{u}_2 = 0$.
   - *iii*) Positive. $0 < \lambda_1 < \lambda_2$.
     Both $\mathbf{u}_1$ and $\mathbf{u}_2 \to \infty$, $|x_2| = C|x_1|^{\lambda_2/\lambda_1}$.
     An *unstable node*.
     Same as *i*), with direction on trajectories reversed, and 1 and 2 switched.

The cases as characterized by D, contd.

2. $D < 0$: Complex eigenvalues $\mu \pm i\omega$.
   $\mathbf{u}_1 = \mathbf{v}_1 e^{\mu t} e^{i\omega}$, $\mathbf{u}_2 = \mathbf{v}_2 e^{\mu t} e^{-i\omega}$.
   - *i*) Real part negative. $\mu < 0$.
     Both $\mathbf{u}_1$ and $\mathbf{u}_2$ go to zero as $t \to \infty$. Solution spirals around origin in inward spiral. A *stable spiral*.
   - *ii*) Real part zero. $\mu = 0$.
     Solution does not have a limit. Circles around the origin.
     A *center*, neutrally stable.
   - *iii*) Real part positive. $\mu > 0$.
     Both $\mathbf{u}_1$ and $\mathbf{u}_2$ go to infinity as $t \to \infty$. Solution spirals around origin in outward spiral.
     An *unstable spiral*.

Note: If $D = 0$ and the matrix is defective, i.e. not diagonalizable, we have a special case that will be discussed later. We will have a stable or unstable *improper node*.
Note: If the matrix is singular, the origin is not the only critical point. Rather, all multiples of the zero eigenvector $\mathbf{v}$ are critical, so the dynamics takes place along lines at an angle to $\mathbf{v}$, i.e. a solution stays on a line.

Stability diagram

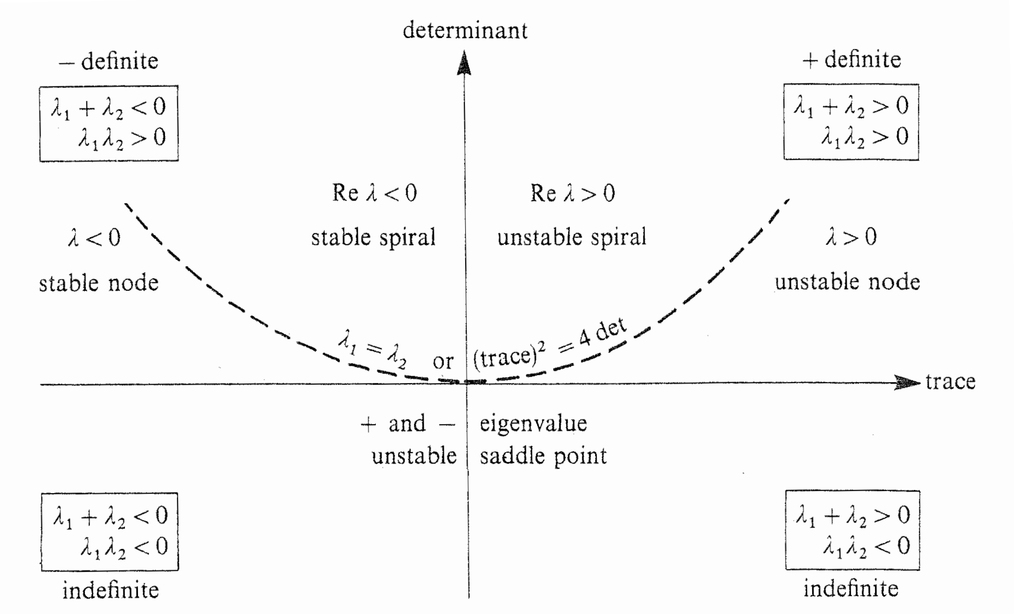


Fig. 6.7. Stability regions for a 2 by 2 matrix.

Solution in the defective case

(defective=non/diagonalizable i.e. not a full set of eigenvectors).
First, let us define the concept of generalized eigenvectors:

Let λ be an eigenvalue of $\mathbf{A}$ with multiplicity m, but with only one eigenvector $\mathbf{x}_1$. Define a sequence of generalized eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ that satisfy

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}_k = \mathbf{x}_{k-1}, \qquad k = 1, \ldots, m,$$

with $\mathbf{x}_0 = 0$.

Suppose that we want to find the general solution of

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u},$$

where the 2×2 matrix $\mathbf{A}$ has a double eigenvalue λ, with only one associated eigenvector $\mathbf{x}_1$.
Let $\mathbf{x}_2$ be a generalized eigenvector of $\mathbf{A}$. The solution is given by

$$\mathbf{u}(t) = (c_1\mathbf{x}_1 + c_2(t\mathbf{x}_1 + \mathbf{x}_2))e^{\lambda t}.$$

Phase portraits and named manifolds

Some curves in phase portraits are more important than others, as they tell us the most about the solution.

This is the case with the so-called *named manifolds*. These are the slow and the fast manifolds of nodes and the stable and unstable manifolds of saddle points.

In the case of a node, which manifold is *slow* and *fast* is determined by the absolute value of λ_1 and λ_2. The one with larger absolute value is the fast one. The manifold is the curve that follows the corresponding eigenvector.