ROYAL INSTITUTE OF TECHNOLOGY

ODE, part 3

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Linearization of nonlinear system

Consider an autonomous system of 2 first order equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0,$$

$\mathbf{u} = (u_1(t), u_2(t))^T$, $\mathbf{f} = (f_1(\mathbf{u}), f_2(\mathbf{u}))^T$.
Assume that $\mathbf{u}^* + \mathbf{v}(t)$ is a solution, then

$$\frac{d}{dt}(\mathbf{u}^* + \mathbf{v}(t)) = \frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{u}^* + \mathbf{v}) = \underbrace{\mathbf{f}(\mathbf{u}^*)}_{=0} + \mathbf{J}(\mathbf{u}^*)\mathbf{v} + O(\|\mathbf{v}\|^2),$$

with the Jacobian $J_{ij} = \frac{\partial f_i(\mathbf{u})}{\partial u_j}$ evaluated at $\mathbf{u}^*$.
$\mathbf{J}(\mathbf{u}^*)$ is a constant matrix, let us denote it by $\mathbf{A}$.

The dynamics close to the critical point is determined by $\mathbf{A}$, and we can learn a lot from studying the behavior of solutions of the linear system

$$\frac{d\mathbf{v}}{dt} = \mathbf{A}\mathbf{v}.$$

Stability of critical points

1. Determine the critical points $\mathbf{u}^*$, where $\mathbf{f}(\mathbf{u}^*) = 0$.
2. Compute the Jacobian

$$\mathbf{A} = \mathbf{J}(\mathbf{u}) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix}$$

   and evaluate it at $\mathbf{u} = \mathbf{u}^*$.
3. Determine the stability or instability of the linearized system by the eigenvalues of **A**.

If $\mathbf{f}(\mathbf{u}^*) = 0$ then for initial values near $\mathbf{u}^*$ the nonlinear equation $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ imitates the linearized equation with matrix $\mathbf{A} = \mathbf{J}(\mathbf{u}^*)$:

if **A** is *unstable* at $\mathbf{u}^*$ then so is $\mathbf{u}' = \mathbf{f}(\mathbf{u})$.

if **A** is *stable* at $\mathbf{u}^*$ then so is $\mathbf{u}' = \mathbf{f}(\mathbf{u})$.

The nonlinear equations has spirals, nodes, and saddle points according to **A**. However, for "borderline" case of a center, the stability of $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ is undecided.

Undecided cases

if **A** has a center (neutral stability with purely imaginary eigenvalues) the stability of $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ is undecided. The nonlinear system can have either a center or a spiral.

if **A** has a borderline node (one double real eigenvalue), the nonlinear system can have either a node or a spiral.

First example: damped pendulum

Non-dimensionalized equation:

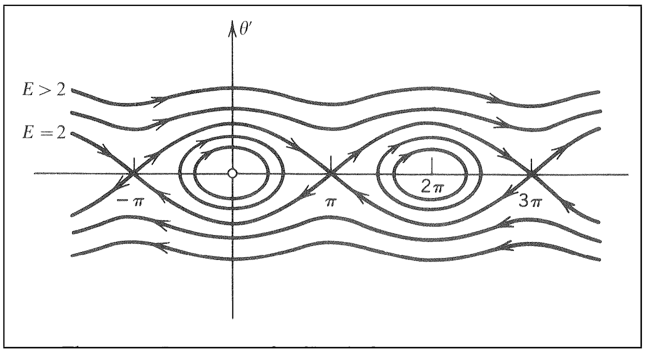
$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \sin\theta = 0.$$

With $u_1 = \theta$, $u_2 = d\theta/dt$, get first order system

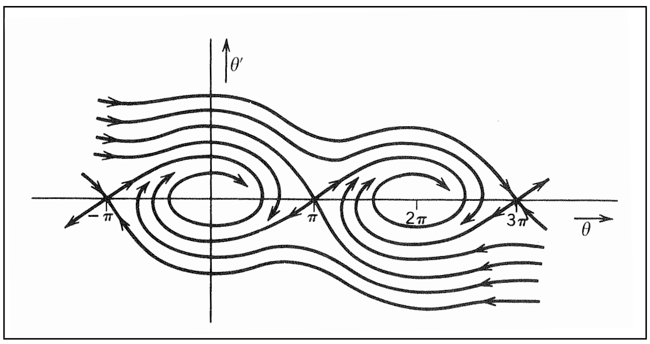
$$\begin{aligned}\frac{du_1}{dt} &= u_2 \\ \frac{du_2}{dt} &= -\sin(u_1) - cu_2\end{aligned}$$

Critical points, $(p\pi, 0)$, p integer. Analysis shows odd multiples of π unstable critical points (mass stationary at top). Stability of critical points with even multiples of π (pendulum hangs down) depends on value of c. [NOTES]

Phase portraits for the pendulum



Undamped pendulum ($c = 0$): Orbits in the phase plane are contours of constant energy.



Damped pendulum ($0 < c < 2$): Curves spiral into equilibrium. For larger damping, picture changes again and the stationary points are nodes.

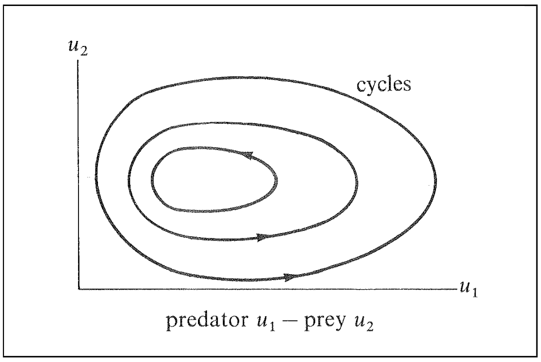
Population dynamics

A simple predator prey model:

$$u_1' = au_1 - bu_1u_2 \quad (1)$$
$$u_2' = cu_1u_2 - du_2, \quad (2)$$

where u_1 represents the population of the prey and u_2 represents the population of the predator, and $a, b, c, d > 0$.

Analysis of linearized system yields one critical point that is a saddle point, and one center. Stability of this last point undecided for nonlinear system. [NOTES]



Periodic solutions, i.e. closed curves:

$$a \log u_2 - bu_2 = cu_1 - d \log u_1 + C,$$

where the constant C is determined by $u_1(0)$ and $u_2(0)$.

Periodic solutions - limit cycles.

- A linear system has closed paths only if the eigenvalues of the system matrix are purely imaginary. In this case, every path is closed.
- A nonlinear system can perfectly well have a closed path that is isolated.
- A "limit cycle" is a periodic orbit that trajectories approach.

The Poincaré-Bendixson theorem

Any orbit of a 2D continuous dynamical system which stays in a closed and bounded subset of the phase plane forever must either tend to a critical point or to a periodic orbit.

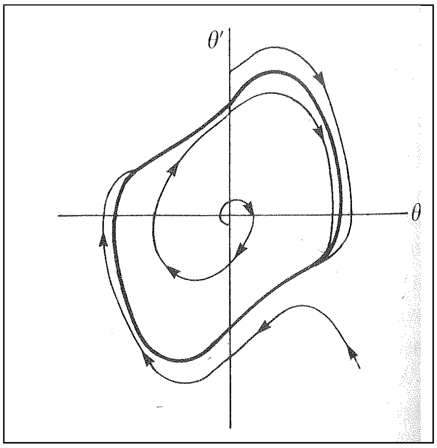
Hence, if this subset of the phase plane has no critical point, the solution must approach a periodic orbit.

There is no such theorem for systems of larger dimension. It is the special topology of the plane that makes the difference: A closed curve (periodic orbit) divides the plane into two disjoint sets, and orbits cannot cross the boundary.

Famous example: Van der Pol equation

$$\frac{d^2\theta}{dt^2} + \mu(\theta^2 - 1)\frac{d\theta}{dt} + \theta = 0, \quad \mu > 0$$

- Again, "damped" oscillations, but not certain that the damping is positive.
- For small θ, the "damping" is negative, and the amplitude grows. For large θ, the damping is positive, and solution decays.
- However, not periodic for all initial values. When the solution leaves a very small or very large value, it can not get back.
- Instead, all orbits spiral toward a *limit cycle* which is the unique periodic solution to van der Pol's equation.



Hopf bifurcation

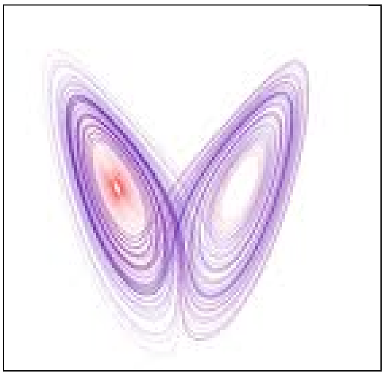
The appearance or the disappearance of a periodic orbit through a local change in the stability properties of a steady point is known as the *Hopf bifurcation*.

Let $J0$ be the Jacobian of a continuous parametric dynamical system evaluated at a steady point $\mathbf{u}^*$ of it.
Suppose that all eigenvalues of $J0$ have negative real parts except one pair of conjugate nonzero purely imaginary eigenvalues $\pm i\beta$.
A Hopf bifurcation can arise when these two eigenvalues crosses the imaginary axis because of a variation of the system parameters.
(More specific conditions exist).

Remark. For linear systems, we can also talk about a bifurcation as stability is lost, i.e. as a pair of eigenvalues "crosses the imaginary axis". This is however not a Hopf bifurcation (can never get a limit cycle).

The Lorenz attractor - The butterfly effect

Simplified model of weather. A 3-system. The equations that govern the Lorenz oscillator are



$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z,$$

where σ is called the Prandtl number and ρ is the Rayleigh number. All $\sigma, \rho, \beta > 0$, but usually $\sigma = 10$, $\beta = 8/3$, and ρ is varied.

Trajectories that starts close by drift apart after a while, and flip from one leaf of the structure to the other, seemingly at random. The system is deterministic, but very sensitive to perturbations - the butterfly effect.