ROYAL INSTITUTE
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Summary

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Mathematical Models, Analysis and Simulation
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Linear Algebra

Read Strang Chapter 1 and Section 2.3.

Matrices

- Range, rank, nullspace, nullity.
- Singular/non-singular matrix. Determinant. Inverse.
- Orthogonal matrix.
- Symmetric positive definite matrix (SPD).
- Permutation matrix to swap rows.
  Permutation matrix to swap columns.

Linear systems of equations

- When does $\mathbf{Ax} = \mathbf{b}$ have a unique solution? No solution? Many solutions?
- Gaussian elimination.
- LU factorization.
- Cholesky factorization.

Eigenvalues, eigenvectors and diagonalization

Eigenvalues and eigenvectors

- Definition of eigenvalues and eigenvectors.
- Relation between eigenvalues of $\mathbf{A}$ and determinant of $\mathbf{A}$, trace of $\mathbf{A}$, eigenvalues of $\mathbf{A}^{-1}$ and in general $\mathbf{A}^p$.
- Multiplicity of eigenvalues.
- Linear independence of eigenvectors.
- How to compute for small systems (by hand).

Diagonalization

- What is needed for a matrix $\mathbf{A}$ to be diagonalizable, i.e. that there exist an invertible matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}$ is diagonal?
- Assume we know the eigenvalues of $\mathbf{A}$, when does one know for sure that $\mathbf{A}$ is diagonalizable? When do we need to check more, and what do we need to check?
- What is special if $\mathbf{A}$ is symmetric?

Vector and Matrix norms and Least squares solutions

Vector and Matrix norms

- Definition of Euclidean norm of vector, 2-norm of matrix.
- What is the Rayleigh quotient?
- How is the condition number of a matrix defined? Why is it important?

Least squares solution

- What is a least squares solution? What quantity does it minimize?
- Give a geometric example for the questions above.
- What are the "normal equations"? The system matrix for the normal equations are SPD. Why?
- For any SPD matrix, relate the solution to the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ to a minimization problem.
- How would you compute a least squares solution?

Algorithms

- What is the Gram-Schmidt orthogonalization? How does it work and what can you use it for?
- Are there alternative algorithms to Gram-Schmidt for doing the same thing?
- What is a QR factorization? When does it exist? Why is it useful?
- How can you compute a QR factorization?
- What do the normal equations become when you have a QR factorization of the system matrix?
- What is a singular value decomposition (SVD)? When does it exist?
- How are the singular values and the eigenvalues of a matrix related?
- How can the singular values tell us the rank of $\mathbf{A}$?
- What is a pseudoinverse of $\mathbf{A}$?
- What is the operation count of an algorithm, and why is it important?
- To pick an algorithm to compute a least squares solution, what properties of the algorithm would you consider in addition to the operation count?

Equilibrium problems

Read Strang, sections 2.1-2.2, 2.4-2.5, 2.7.
Handout about Modified Nodal Analysis by M. Hanke.

A system in equilibrium, e.g. a mechanical or electrical system. The equilibrium solution is the solution that minimizes the energy of the system.

Graph models

- What is the incidence matrix?
- Assume n nodes and m edges in the graph. What will be the size of the incidence matrix?
- How does the incidence matrix relate node values and differences across edges?
- What does it mean to "ground a node"? Why is it needed?

Equilibrium problems

Assume n nodes and m edges in the graph.
Let $\mathbf{u}$ be the vector of node values (displacements or potentials),
$\mathbf{w}$ the vector of internal forces or currents, and $\mathbf{A}$ the incidence matrix.
Let $\mathbf{e}$ be the elongation of springs in a mechanical system, and the voltage drop over an edge in an electric system.

Mechanical systems and Electrical systems

- Express the condition that the forces should be in equilibrium in a mechanical system using the incidence matrix $\mathbf{A}$. Express Kirchoff's voltage law using the incidence matrix $\mathbf{A}$.
- Express Hooke's law and Ohm's law in matrix form.
- Express the relation between elongation of springs and displacements of masses, and the relation between voltage drops and node voltages, using the matrix $\mathbf{A}$.
- For both cases, put it together to a system of equations. What is the stiffness matrix?

Equilibrium problems - Trusses

- What is the incidence matrix for trusses?
- Express the *linearized* relation between displacements and stretching. What assumptions have been made to obtain this relation?
- What can we learn from the incidence matrix about the stability of a truss? When is it stable, when is it unstable?

Electrical circuits

- Write down Kirchoff's voltage law. Show that it is automatically satisfied using the relation between node voltages and voltage drops over branches.
- How is the system modified if voltage sources are added to the system, where do they enter? Same question for current sources?

Modified Nodal Analysis (MNA)

- In MNA, incidence matrices are transposed and their signs are changed. Also, an incidence matrix is written down for every "type" of branch.
- Starting with Kirchoff's current law, assuming resistive branches and branches with current and voltage sources, derive the MNA equations.

Modified Nodal Analysis (MNA), contd

- Assume a sinusoidal forcing term of fixed frequency ω. Assume that there are resistors, inductors and capacitors in the circuits. Using $j\omega$ analysis, write down the algebraic law relating voltage and current amplitudes. Starting from Kirchoff's current law, derive the system of equations to be solved.
- When there is not a single frequency, transient analysis must be used. Again, starting with Kirchoff's current law, derive the system of equations to be solved. Assuming a nonlinear conductance (see HW 4), how must the equations be modified?
- Work on different examples of circuits, and make sure that you know how to create incidence matrices and set up the full system.

Constrained optimization

Section 8.1 of Strang.

- What is a Lagrange multiplier? How can it be used when optimizing with constraints?
- See Exam from January 2006, problem 6. Work through this problem. (Available on homepage with solution).
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Ordinary differential equations - dynamic systems

Dynamic problems. Example: a line of springs.

- Now, not looking for equilibrium, but the evolution in time.
- The "law of nature" is now Newton's law instead of requiring that forces are in equilibrium. I.e. use the fact that a net force will give rise to an acceleration.
- With this modification, derive the system of equations.
- Consider also the case of a damped system, introducing a force proportional to velocity, with a damping coefficient.
- Make sure you know how to non-dimensionalize the equation. Compare HW and lecture notes.

System of first order ODEs

Strang section 2.2.

- A second order ODE can be rewritten as a system of first order ODEs. Make sure you know how.
- This technique generalizes also to higher order equations by introducing new variables for the derivatives.
- What does it mean that the system is written on autonomous form?
- The general solution of a system of first order ODEs can be written in terms of the eigenvalues and eigenvectors of the system matrix.
- The λ_is determine which modes that will grow and which modes that will decay.

Numerical methods for ODEs

- For numerical stability, consider the model equation:

$$u_t = \lambda u, \quad u(0) = u_0.$$

- What is the stability region for a numerical method? Knowing this, how can you pick a time-step to compute for a system of first order ODEs?
- What is the stability region for forward Euler? For backward Euler?
- In what cases would you choose an implicit method?
- What is meant by the order of accuracy of a method? Given a numerical algorithm as a "black box", how would you determine what order of accuracy it has?

Linear constant coefficient 2D systems

Very little material in Strang. Material about Phase planes can be found in e.g. Strangs "old book", "Introduction to Applied Mathematics", or any basic text on ODEs, e.g. "Differen tial equations with applications and historical notes" by Strauss.

Consider

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}.$$

Eigenvalues of $\mathbf{A}$, λ_1 and λ_2. Corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

- What is the general solution for diagonalizable systems?
- What is the solution in the defective case?

Phase portraits

- What is a phase portrait?
- What is a node, a spiral , a saddle point and a center?
- What is a named manifold? For which of the four types of points above do we have named manifolds?
- In which case do we have a slow and a fast manifold? In which case do we have one stable and one unstable manifold?
- Assume that we have two real and distinct eigenvalues. What types of critical points are possibe?
- Same question for complex valued eigenvalues.
- Same question for a double real eigenvalue.

Critical points and linearized equations

Consider a system of n first order equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u}(0) = \mathbf{u}_0,$$

$\mathbf{u} = (u_1(t), \ldots, u_n(t))^T$, $\mathbf{f} = (f_1(\mathbf{u}, t), \ldots, f_n(\mathbf{u}, t))^T$.
Here, $\mathbf{f}$ might be a nonlinear function of $\mathbf{u}$.

- What is a critical point?
- What is the Jacobian?
- It is in general difficult to analyze the behavior of a non-linear system. Which linear system is of interest to study and why?
- Will the linear analysis always determine the stability of the critical points? What happens in a borderline case, such as if the linear analysis yields that the critical point is a center?
- What is a limit cycle? Read the Poincaré-Bendixson theorem.

PDEs

- Three main classes of well-posed PDEs: Elliptic, parabolic and hyperbolic.
- For each class above: What kind of phenomena are modeled? Give a model equation.
- What are the requirements for a PDE to be well posed?
- For model equations, should be able to show well posedness.

Elliptic equations

Strang, Chapter 3.

- The Laplace and Poisson equations.
- What is a harmonic function?
- What does the maximum principle say?
- Show uniqueness of solution to Poisson equation with Dirichlet boundary conditions.
- What is the fundamental solution for Laplace's equation?
- The separation of variables technique can be applied to solve Laplace's equation in simple geometries. Make sure you know how!

Finite difference methods

- Apply a second order finite difference method to Poisson's equation in 1D and 2D. What is the structure of the matrix in each of these two cases?
- What is the local truncation error?
- Why is it not sufficient to know the local truncation error to determine the error in the solution? What else do we need to consider?

Finite element methods

- How are the weak form of the equations obtained for an elliptic PDE?
- How and when can this be related to a minimization problem?
- Describe Galerkin's method.
- Pick an elliptic equation and show that the residual to the Galerkin solution is orthogonal to the test space.
- Apply Galerkin's method to a specific elliptic equation, with a specific choice of basis functions. What is the structure of the resulting matrix?
- How do you deal with inhomogeneous Dirichlet boundary conditions, i.e. BC where the function is required to take values different from 0 at the boundaries?
- What would you say is the main advantage of finite element methods as compared to finite difference methods?

Parabolic equations

- Show well posedness of the heat/diffusion equation.
- What is the diffusion kernel? What does the solution formula, as expressed in integral form with the diffusion kernel tell us about the solution?
- How do you discretize a parabolic PDE using the "method of lines"? How can the stability condition be determined using the stability region of the ODE method that is used?
- What is the CFL number? What is the typical scaling for an explicit in time solver for a parabolic problem?
- What is the Lax equivalence theorem? Why is it important?
- Define consistency and stability of a numerical method.
- How do you in practice show that a method is consistent?
- How can you show stability (and obtain stability conditions) using the von Neumann (Fourier) analysis.

Hyperbolic equations

- Show well posedness of the advection equation.
- How are the characteristics, or characteristic lines defined? Show that the solution is constant along characteristics.
- How and with what speed does information propagate in hyperbolic problems?
- How does the concept of characteristics etc generalize to systems of hyperbolic PDEs?
- Define inflow and outflow boundaries.
- Where should we set boundary conditions? How many boundary conditions can we set?
- Rewrite the wave equation as a 2×2 system with only first derivatives in time. Discuss boundary conditions in this case.

Numerical methods for hyperbolic equations

- Give example of some methods that can be used to discretize the advection equation.
- What does it mean that a numerical method is conservative?
- What is the "domain of dependence" and "the general CFL condition"?
- Certainly, the Lax Equivalence theorem can be used here too.
- Pick a method and show consistency and also order of the method by finding the local truncation error, and find the stability requirement using von Neumann analysis.
- What is the typical scaling of Δt as related to Δx in a hyperbolic problem? Comment on the use of implicit time-stepping methods.
- What are dissipative and dispersive errors? Consider the propagation of a square pulse. Draw a typical solution with dissipative error, then one with dispersive error.
- What can we learn from the modified equations?

Fourier series

Section 4.1 in Strang.

- Consider the "classical" Fourier series, either written with sin and cos terms, or complex exponentials. How can the coefficients of these series be computed? What is the orthogonality condition that is used to obtain this formula?
- In applying the separation of variables technique, we can get other Fourier series, depending on the boundary conditions of the problem. What is symmetric boundary conditions, and what do they guarantee? Write a Fourier series in general terms, without specifying the form of the eigenfunctions, and write down the definition of the coefficients.
- Give some different notions of convergence. Which one is the strongest?
- What is Gibbs' phenomenon?
- What is Parseval's identity?
- How is the decay of Fourier coefficients related to the regularity of the function that has been expanded?

The discrete Fourier transform and the FFT

Section 4.3 in Strang.

- What is the discrete and the inverse discrete transform? What do you get if you apply the discrete transform on a set of point values f_j $j = 0, \ldots, N-1$ and then the inverse discrete transform to the result?
- How costly would it be to evaluate the DFT directly?
- Describe the idea behind the FFT. What is the computational complexity of this algorithm?
- How do the coefficients as obtained by the DFT relate to the Fourier coefficients? What are aliasing errors?

Spectral interpolation and differentiation

- Describe the idea of spectral interpolation.
- What kind of convergence would you expect, and how does this depend on the function that you want to interpolate? (HW5).
- Describe the idea of spectral differentiation.
- What kind of convergence would you expect, and how does this depend on the function that you want to differentiate? (HW5).

Spectral methods for differential equations

- Consider a time-dependent linear PDE with peridodic boundary conditions and derive the ODEs for the time dependent Fourier coefficients using either a Galerkin or collocation approach (they yield the same result).
- Discuss the errors of such a method.
- Next, consider a non-linear PDE such as the Burger's equation. Use Galerkin's method to derive a spectral method.
- Discuss the treatment of the non-linear term. Cost of full evaluation.
- Consider a pseudo-spectral treatment of the non-linear term. Describe how it is done. What are aliasing errors? How can they be removed?
- Write down a collocation method for Burger's equation.
- Make sure to get familiar with the generalization to two and three dimensional problems. (Straight forward). Consider e.g. the example on incompressible Navier-Stokes equations covered in class.

Fourier integrals and the sampling theorem

Strang, section 4.5, p367-363 and p 691-693.

- The Fourier transform defines a *continuous* k-spectrum in Fourier space. What is Plancherel's theorem? What is the "discrete" counterpart?
- What is meant by a "band-limited" function?
- What is the Nyquist sampling rate?
- The sampling theorem describes two processes in signal processing: a sampling process, in which a continuous time signal is converted to a discrete time signal, and a reconstruction process, in which the original continuous signal is recovered from the discrete time signal. What is the result of the sampling theorem?

The sampling theorem

This is how the sampling theorem is described on Wikipedia (more details available online):

- *The sampling process:* The continuous signal varies over time and the sampling process is performed by measuring the continuous signal's value every T units of time, which is called the sampling interval. This results in a sequence of numbers, called samples, to represent the original signal.
- *The reconstruction:* Each sample value is multiplied by the sinc function scaled so that the zero-crossings of the sinc function occur at the sampling instants and that the sinc function's central point is shifted to the time of that sample, nT. All of these shifted and scaled functions are then added together to recover the original signal. The scaled and time-shifted sinc functions are continuous making the sum of these also continuous, so the result of this operation is a continuous signal.
- *The condition:* The signal obtained from this reconstruction process can have no frequencies higher than one-half the sampling frequency. According to the theorem, the reconstructed signal will match the original signal provided that the original signal contains no frequencies at or above this limit. This condition is called the Nyquist criterion.