# Systems in equilibrium.
# Graph models and Kirchoff's laws.

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## A system in equilibrium

[Material from Strang, sections 2.1 and 2.2].
Consider a system of springs and masses in equilibrium.
[NOTES]

To obtain our system of equations, we have applied three equations:

1) The forces should be in equilibrium.
2) Hooke's law for springs.
3) Relation between spring length and mass positions $x_i$.

The system is on the form: $\mathbf{A} = \begin{bmatrix} -\mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix}$, where the columns of $\mathbf{A}$ are linearly independent, and $\mathbf{C}$ is a diagonal matrix with positive entries on the diagonal.

This yields $\mathbf{A}^T\mathbf{C}\mathbf{A}\mathbf{x} = \mathbf{f} + \mathbf{A}^T\mathbf{C}\mathbf{b}$.
"Stiffness matrix" $\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A}$ is symmetric positive definite.
Strang works with displacements $u_i$ instead of positions $x_i$. In this case, replacing $\mathbf{x}$ by $\mathbf{u}$ in the system, we will have $\mathbf{b} = 0$. [NOTES]

## Minimization

We have

$$\mathbf{Ku} = \mathbf{f}, \text{ with } \mathbf{K} = \mathbf{A}^T\mathbf{CA}$$

From earlier:
The solution to $\mathbf{Ku} = \mathbf{f}$ is the $\mathbf{u}$ that minimizes the quadratic form $P(\mathbf{u}) = 1/2\mathbf{u}^T\mathbf{Ku} - \mathbf{u}^T\mathbf{f}$.

- Stretching increases the internal energy, $1/2\mathbf{e}^T\mathbf{Ce} = 1/2\mathbf{u}^T\mathbf{Ku}$. ($\mathbf{e}$ is the elongation of the spring).
- The masses loose potential energy by $\mathbf{f}^T\mathbf{u}$ (force times displacement, work done by external forces).

This is NOT a conserved system: external forces (in our example gravitational forces) are active.

Hence, the configuration of this system at equilibrium is the configuration that minimizes the potential energy of the system.

## Incidence matrix A

$\mathbf{A}$ is the so called "incidence matrix". [NOTES - EXAMPLES]

In our example, the structure was simply a line. Can have a "graph" instead. Will develop these graph models further and apply it to electric circuits. (Strang, Section 2.3).

Numbers $u_i$ can represent the heights of the nodes, the pressure at the nodes, or the voltages at the nodes, dependent on the applications. Common language: "potentials".
$\mathbf{Au}$: the "potential difference", i.e the difference across an edge.

The nullspace of $\mathbf{A}$ contains the vector that solve $\mathbf{Au} = 0$. If we do not set any potential $u_i$, the nullspace is a line. The columns of $\mathbf{A}$ is a line, it has dimension 1. The columns of $\mathbf{A}$ are linearly dependent and $\mathbf{A}^T\mathbf{A}$ is singular.

In our example with masses, we fixed one location. For electrical circuits, we say that we "ground one node". This will remove one column of $\mathbf{A}$, and the remaining columns will be linearly independent. Then $\mathbf{A}^T\mathbf{A}$ is invertible.

## The graph Laplacian matrix

The matrix $\mathbf{A}^T\mathbf{A}$ turns out to have a very specific form:

- On the diagonal:

$$(\mathbf{A}^T\mathbf{A})_{jj} = \text{degree} = \text{number of edges meeting at node } j.$$

- Off the diagonal:

$$(\mathbf{A}^T\mathbf{A})_{jk} = \begin{cases} -1 & \text{if nodes } j \text{ and} k \text{ share an edge.} \\ 0 & \text{if no edge goes between nodes } j \text{ and } k. \end{cases}$$

And for $\mathbf{A}^T\mathbf{C}\mathbf{A}$:

- On the diagonal changes to $(\mathbf{A}^T\mathbf{C}\mathbf{A})_{jj} =$ sum of all spring constants/conductivities/... in edges meeting at node $j$.
- Off the diagonal: the $-1$ changes to the negative of the spring constant/conductivity/... in the branch between node $j$ and $k$.

## Kirchoff's laws

Consider one node with brach currents $w_1, \ldots, w_l$ entering this node.

- Kirchoff's current law (KCL): $w_1 + \cdots + w_l = 0$.

Sum of all branch current entering a node equals zero.

Consider one loop with brach voltages (or voltage drops) $e_1, \ldots, e_n$.

- Kirchoff's voltage law (KVL): $e_1 + \cdots + e_n = 0$.

Sum of all branch voltages in a loop equals zero.

In a practical network, many nodes and loops. Need a systematic way to derive all these equations for a given network.

## Equations

Kirchoff's current law (KCL) yields

$$\mathbf{A}^T\mathbf{w} = \mathbf{0}.$$

Kirchoff's voltage law is automatically satisified with

$$\mathbf{e} = -\mathbf{A}\mathbf{u}.$$

Ohm's law yields

$$\mathbf{w} = \mathbf{C}\mathbf{e},$$

where $\mathbf{C}$ is a diagonal matrix with entries $c_i > 0$, where $c_i$ is the conductance in branch $i$.
(have $c_i = 1/R_i$, where $R_i$ is the resistance value. $e_i = R_i w_i$).

- $m$ edges, $n$ nodes. Ground one node. $\mathbf{A}$ is $m \times (n-1)$.
- Node potentials $\mathbf{u}$, branch currents $\mathbf{w}$, branch voltages (or voltage drops) $\mathbf{e}$,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_{n-1} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}$$

## Adding current and voltage sources

Adding voltage sources and current sources to the system, yields the modifications:

- $\mathbf{A}^T\mathbf{w} = \mathbf{f}$ (current source in $\mathbf{f}$) ,
- $\mathbf{e} = \mathbf{b} - \mathbf{A}\mathbf{u}$, (voltage source in $\mathbf{b}$).
- Ohm's law, $\mathbf{w} = \mathbf{C}\mathbf{e}$ remains the same.

As a large system, this reads:

$$\begin{bmatrix} \mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix}.$$

where $\mathbf{C}$ is a diagonal matrix with positive entries on the diagonal (the conductances). $\mathbf{A}$ is the incidence matrix.
This yields

$$\mathbf{A}^T\mathbf{C}\mathbf{A}\mathbf{u} = \mathbf{A}^T\mathbf{C}\mathbf{b} - \mathbf{f}.$$

"Stiffness matrix" $\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A}$ is symmetric positive definite if the columns of $\mathbf{A}$ are linearly independent - "ground a node".

# Modified Nodal Analysis (MNA)

It is common in engineering context to work systematically with incidence matrices also for voltage source and current source branches.
We will denote

- $\mathbf{A}_R = \mathbf{A}$, the incidence matrix of the resistive branches.
- $\mathbf{A}_V$ the incidence matrix for all voltage source branches.
- $\mathbf{A}_I$ the incidence matrix for all current source branches.
- Assume $N_R$ resistive branches, $N_V$ voltage source branches, $N_I$ current source branches.
  With $m$ nodes, $\mathbf{A}_R$ is $N_R \times m$, $\mathbf{A}_V$ is $N_V \times m$, $\mathbf{A}_I$ is $N_I \times m$.
- Let
  $\mathbf{i}_v$: branch currents trough voltage sources $(N_V \times 1)$.
  $\mathbf{I}$: vector of values of all current sources $(N_I \times 1)$.
  $\mathbf{E}$: vector of values of all voltage sources $(N_V \times 1)$.

The notation we will use differs slightly from common MNA notation (mainly s.t. $\mathbf{A}_R^{MNA} = -\mathbf{A}_R^T$ etc).
Choosing our notation to stay closer to the notation of Strang.

# Modified Nodal Analysis (MNA), contd

We obtain the system

$$\begin{bmatrix} \mathbf{A}_R^T \mathbf{C} \mathbf{A}_R & -\mathbf{A}_V^T \\ -\mathbf{A}_V & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{i}_V \end{bmatrix} = \begin{bmatrix} \mathbf{A}_I^T \mathbf{I} \\ \mathbf{E} \end{bmatrix}.$$

- $\mathbf{A}_R$, $\mathbf{A}_V$ and $\mathbf{A}_I$ incidence matrices for resistive, voltage source and current source branches, respectively.
- $\mathbf{C}$ diagonal matrix with conductances (inverse of resistances).
- $\mathbf{I}$: vector of values of all current sources, $\mathbf{E}$: vector of values of all voltage sources.
- $\mathbf{u}$: vector of node potentials, $\mathbf{i}_v$: branch currents trough voltage sources.

NOTES:

- How to get to this system.
- Example earlier done with Strang's notation in this notation.

# RLC circuits and AC analysis

R - resistor, L - inductor, C - capacitor.
Voltage drops $V$, current $I$. (Strang's notation).
Characteristic equations:

$$V = RI, \quad I = C\frac{dV}{dt}, \quad V = L\frac{dI}{dt}$$

(if capacitance value $C$ and self-inductance value $L$ are constant).

Assume sinusoidal forcing term of fixed frequency $\omega$.
Typical voltage $V(t) = \hat{V}\cos(\omega t) = \Re(\hat{V}e^{j\omega t})$. Current $I(t) = \Re(\hat{I}e^{j\omega t})$.
Since

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t},$$

we get the simple algebraic law

$$\hat{V} = Z\hat{I},$$

with $Z = 1/R$ for resistors, $Z = j\omega L$ for inductors and $Z = 1/(j\omega C)$ for capacitors.

# $j\omega$ analysis - Strang's approach

Same system as before:

$$\begin{bmatrix} \mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix}.$$

But now, the diagonal matrix $\mathbf{C}$ contains the complex impedances $Z$.

Consider an example where branch 1 has a resistor with resistance $R_1$, branch 2 a inductor with self-inductance $L_2$, branch 3 a capacitor with capacitance $C_3$ and finally, branch 4 a resistor with resistance $R_4$.
Then:

$$\mathbf{C} = \begin{bmatrix} 1/R_1 & & & \\ & 1/(j\omega L_2) & & \\ & & j\omega C_3 & \\ & & & 1/R_4 \end{bmatrix}.$$

# $j\omega$ analysis by MNA

Before, we had incidence matrices $\mathbf{A}_R$, $\mathbf{A}_V$ and $\mathbf{A}_I$, for resistive branches, voltage source branches and current source branches, respectively.
Now, introduce also the incidence matrix $\mathbf{A}_C$ for capacitive branches, and $\mathbf{A}_L$ for inductive branches.

The admittances $Y$ are the inverses of the impedances $Z$.
Introduce the diagonal matrices $\mathbf{Y}_R$, $\mathbf{Y}_C$ and $\mathbf{Y}_L$ that contain the admittances of the resistive, capacitive and inductive branches, respectively.
(Before, $\mathbf{Y}_R = \mathbf{C}$).

The system can now be written as

$$\begin{bmatrix} \mathbf{A}_R^T\mathbf{Y}_R\mathbf{A}_R + \mathbf{A}_C^T\mathbf{Y}_C\mathbf{A}_C + \mathbf{A}_L^T\mathbf{Y}_L\mathbf{A}_L & -\mathbf{A}_V^T \\ -\mathbf{A}_V & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{i}_V \end{bmatrix} = \begin{bmatrix} \mathbf{A}_I^T\mathbf{I} \\ \mathbf{E} \end{bmatrix}.$$

(Derivation in NOTES on board).