**HOMEWORK 1**
**for Mathmatical Models, Analysis and Simulation, Fall 2011**
Report due Mon Sep 19, 2011.
Maximum score 6.0 pts.

Read chapter 1 of Strang.
Solve the following problems:

1) Consider the $n \times n$ matrix $\mathbf{A}$ with the elements

$$a_{ii} = \alpha, \quad i = 1, \ldots, n, \qquad a_{ij} = 1, \quad i \neq j.$$

   a) What is the rank of $\mathbf{A}$? (Note that this depends on $\alpha$).
   b) What are the eigenvalues of $\mathbf{A}$?
   c) For which values of $\alpha$ is $\mathbf{A}$ positive definite?

2) $\mathbf{A}$ is a real symmetric $n \times n$ matrix with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The following iterative scheme is used to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$,

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha \mathbf{r}_k, \quad \mathbf{r}_k = \mathbf{A}\mathbf{x}_k - \mathbf{b}, \quad k = 0, 1, \ldots$$

given an initial guess $\mathbf{x}_0$.
Let $\mathbf{e}_k = \mathbf{A}^{-1}\mathbf{b} - \mathbf{x}_k$ and $\kappa = \lambda_n / \lambda_1$, the condition number of $A$.

   a) Show that $\mathbf{e}_{k+1} = \mathbf{B}\mathbf{e}_k$, with $\mathbf{B} = \mathbf{I} - \alpha \mathbf{A}$. What are the eigenvalues of $\mathbf{B}$?
   b) Show that

$$\frac{\mathbf{e}_{k+1}^T \mathbf{e}_{k+1}}{\mathbf{e}_k^T \mathbf{e}_k} \leq \max_{1 \leq l \leq n} (1 - \alpha \lambda_l)^2.$$

   c) Show that one can choose $\alpha$ such that

$$\|\mathbf{e}_{k+1}\|^2 \leq \frac{\kappa - 1}{\kappa + 1} \|\mathbf{e}_k\|^2.$$

3) How closely, as measured by the $L^2$ norm on the interval $[1, 2]$, can the function $f(x) = x^{-1}$ be fitted by a linear combination of the functions $e^x$, $\sin x$ and $\Gamma(x)$? ($\Gamma(x)$ is the gamma function, and is available as a built-in function in Matlab).

   Write a program that determines the answer with at least 3-digits of accuracy using a discretization of $[1, 2]$ and a discrete least squares problem. Give this answer, as well as the coefficients of the optimal linear combination. Plot the approximation and the error in the approximation.

   (Remember: A discrete approximation of the $L^2$ norm is scaled such that the norm of `ones(1,n)` is independent of $n$).

   (From Trefethen and Bau).

4) Define $\mathbf{t}$ to be a vector of $m$ uniformly spaced points from 0 to 1. Define $\mathbf{A}$ to be the $m \times n$ matrix associated with least squares fitting of a polynomial of degree $n-1$ using the $m$ data points $(t_i, f(t_i))$, $i = 1, \ldots, m$.

   Let $f(t) = \cos(4t)$ and $m = 50$, $n = 12$. Calculate the least squares coefficient vector by four different methods:

   a) Formation and solution of the normal equations, using Matlab's \.
   b) QR factorization computed by Matlab's `qr`.
   c) SVD factorization computed by Matlab's `svd`.
   d) $\mathbf{x} = \mathbf{A}\backslash\mathbf{b}$ in Matlab. What method does this command use?

   Write out the twelve coefficients for the four methods (with 16 digits). Comment on the differences you observe. Which method produces the worst results? Why?

   *Hints for computing:* The matrix $\mathbf{A}$ can be set up in many different ways, one compact way of doing it is to make use of the commands `vander` and `fliplr` in a suitable way. Test your code first on a small problem like $n = 4$, $m = 8$.

   (From Trefethen and Bau).

5) In this exercise we will use the SVD for image compression. Go to the course homepage and download `basketpic.mat`. Load it into matlab.

   This is a $256 \times 256$ matrix. It contains integers between 0 and 255, building up a gray scale picture. To view it, do:
   `imagesc(basketpic);`
   and possibly, if it displays in color:
   `colormap(gray);`
   Now, convert it into a matrix of type double:
   `Amat=double(basketpic);`

   i) Do a singular value decomposition of the matrix, using Matlab's `svd` command. ($\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$)
   ii) Define the Frobenius norm of a matrix:

   $$\|A\|_{Fro} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |A_{ij}|^2}.$$

   Verify that for our matrix (to some precision in Matlab):

   $$\|A\|_{Fro} = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

   with $n = 256$. Show that this is true for a general $m \times n$ matrix $\mathbf{A}$.
   (Hint: What is the trace of $\mathbf{A}^T\mathbf{A}$?).

iii) Plot the singular values against the position in the diagonal of $\mathbf{\Sigma}$ (i.e. your $x$-axis should include $i = 1, 2, \ldots, 256$). Use the `semilogy` command to get a log-scale on the $y$-axis.

iv) Define $\mathbf{A}_k$ to be the rank $k$ approximation of $\mathbf{A}$, i.e.

$$\mathbf{A}_k = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \ldots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

Define $\mathbf{A}_k$ for different values of $k$, and use the commands above to view the resulting image. How large must the rank $k$ be before the image is almost perfect to your eyes?

v) Define the relative error in the Frobenius norm:

$$E_k = \frac{\|\mathbf{A}_k - \mathbf{A}\|}{\|\mathbf{A}\|}$$

Pick 4 different values of $k$. Plot the resulting images for these four values of $k$ in one figure using Matlab's `subplot` command. For each of them, display $E_k$.