HOMEWORK 4: Numerical solution of PDEs
for Mathmatical Models, Analysis and Simulation, Fall 2011
Report due Mon Nov 14, 2011.
Maximum score 6.0 pts.

Read Strang's (old) book, Sections 3.1-3.4 and 6.4-6.5.

1. Selected theoretical problems (Score: 1.5)

a) If $u(x,y) = x^2$ in the square $S = \{-1 < x, y < 1\}$, compute both sides of

$$\iint_S \nabla \cdot (\nabla u)\, dxdy = \int_C \hat{\mathbf{n}} \cdot \nabla u\, ds. \quad \text{(Divergence theorem)}$$

b) If $curl\,\mathbf{v} = 0$ and $div\,\mathbf{w} = 0$ in a three-dimensional volume V, with $\mathbf{w} \cdot \hat{\mathbf{n}} = 0$ on the boundary, show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal,

$$\iiint \mathbf{v}^T \mathbf{w} dV = 0.$$

How could an arbitrary vector field $\mathbf{f}(x,y,z)$ be split into $\mathbf{v} + \mathbf{w}$?

c) Find the coefficients that make $AU_{n+1} + BU_n + CU_{n-1} = Df_n + Ef_{n-1}$ third order accurate for $u' = f(u) = au$. But show that $Az^2 + Bz + C = 0$ has a root with $|z| > 1$, which means exponential instability.

d) Substitute the true $u(x,t)$ into the Lax-Friedrichs method for $u_t = cu_x$. Use $u_t = cu_x$ and $u_{tt} = c^2 u_{xx}$ (constant c) to find the coefficient of the *numerical dissipation* u_{xx}.

e) Consider the Poisson equation with Robin boundary conditions:

$$\Delta u = f \quad \mathbf{x} \in \Omega \subset \mathbb{R}^3$$
$$\frac{\partial u}{\partial n} + a(\mathbf{x})u(\mathbf{x}) = 0 \qquad \mathbf{x} \in \partial\Omega,$$

where $a(\mathbf{x}) > 0$ on $\partial\Omega$. Prove that the solution is unique. (*Hint:* Use Green's first identity.)

2. The convection diffusion equation (Score: 2.0)

Consider the convection-diffusion equation

$$u_t + au_x = \eta u_{xx}, \quad -\infty < x < \infty$$

with $u(x,0) = f(x)$.

Consider either the case with 2π periodic initial data, $f(x) = f(2\pi + x)$, and hence a 2π-periodic solution, or the Cauchy problem, whatever you prefer.

Assume a step size $\Delta x > 0$ and a time step $\Delta t > 0$. The following finite difference approximation is given:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{(2\Delta x)} = \eta\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2}$$

a) What is the order in space and time of the proposed method?

b) What is the stability condition for this method?

   What is the stability condition for the cases $i)$ $a = 0$, and $ii)$ $\eta = 0$?

c) With $a = 1$, for what choices of $\eta > 0$ would you expect this method to work well, and produce an accurate solution? For what choices would you expect it to do worse? (Δt kept within stability limits). Pick some initial conditions and perform numerical computations to illustrate your point. Compare to the solutions obtained when discretizing by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = \eta\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2}$$

   Explain.

3. The wave equation (Score: 2.5)

In this exercise, we will consider the wave equation in one dimension:

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad 0 < x < 1, \\ u(0,x) &= u_0(x), \\ u_t(0,x) &= v_0(x). \end{aligned} \tag{1}$$

The Leap-Frog method for this equation reads:

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = \lambda^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \qquad \lambda = \frac{\Delta t}{\Delta x}, \tag{2}$$

where $u_j^n \approx u(x_j, n\Delta t)$, $x_j = j\Delta x$, $j = 1, \ldots N-1$ and $\Delta x = 1/N$.

a) Implement the Leap-Frog method with Dirichlet boundary conditions, $u(t,0) = u(t,1) = 0$. The Leap-Frog method can then be viewed as solving the system of ODEs

$$\mathbf{u}_{tt} = \mathbf{A}\mathbf{u}, \tag{3}$$

   with central differencing in time and A given by (2). $\mathbf{A}$ will be tridiagonal with $-2/(\Delta x)^2$ on the diagonal and $1/(\Delta x)^2$ on the super and sub diagonals.

Use $N = 64$, $\lambda = 1$ and a pulse as initial data, $u_j^0 = u_0(x_j)$, where

$$u_0(x) = \exp\Big(-100(x-x_s)^2\Big), \qquad x_s = 0.5. \tag{4}$$

Since the Leap-Frog method is a two-step method you need one more initial value to get started. Use the exact solution $u_j^{-1} = [u_0(x_j - \Delta t) + u_0(x_j + \Delta t)]/2$.

Plot the solution in every time step inside your time stepping loop, so that you can see what happens. (You need to add `drawnow` after the plot command for this to work). What happens as the pulses hit the boundary?

Use `subplot` and plot snapshots of the solution at several different points in time in the same figure. Start at $t = 0$ and end at $t = 2$.

Verify that the CFL condition is necessary by varying λ.

b) A flute can be modeled with a version of the wave equation. Assume that the length of the flute is L and that the mean difference in pressure to the ambient pressure in a cross section of the flute is denoted $p(\tau,\xi)$, where τ denotes the physical time and ξ the distance from the mouth piece. There are holes along the flute of different sizes that can be covered or open. A very simple model is that p satisfies the equation

$$p_{\tau\tau} - c_0^2 p_{\xi\xi} + D(\xi)p = G(\tau,\xi), \qquad p(\tau,0) = p(\tau,L) = 0, \tag{5}$$

where c_0 is the speed of sound in air (344 m/s) and $D(\xi)$ is a variable coefficient that is zero where there is no hole and positive at each hole. The larger the whole is, the larger interval where $D(\xi) > 0$. The function $G(\tau,\xi)$ represents the very complicated process of blowing into the mouth piece.

First assume that all holes are covered, such that $D(\xi) \equiv 0$. The root note of the flute is then given by the lowest eigenmode of the system, i.e. the standing wave with the lowest frequency. In our case we have Dirichlet boundary conditions, and $f_{\text{fund}} = c_0/2L$. The lowest note that one can play at a recorder (or English flute) of type C5 (Swedish: Blockflöjt C5) is about 523 Hz, which yields $L \approx 33$ cm. (In reality it is a bit shorter).

If the holes for the fingers are large, the effect of an open hole is the same as if the flute would end at the whole: the pressure in the flute cannot vary much as compared to the surrounding air and p in our equation must be 0 at the hole, i.e. the Dirichlet boundary condition can be placed at the location of the hole instead of at $\xi = L$. If we want to be able to play the frequency f_h we can place a large finger hole at the distance $\hat{\xi} = c_0/2f_h$ from the mouth piece.

Assume that we want to play all tempered notes in an octave above C5 on a recorder, i.e the frequencies $f_k = 2^{k/12} f_0$, med $k = 0,\dots,12$ and $f_0 = 523.25$. We can then place large holes at the distances

$$\hat{\xi}_k \approx (16.4,\ 17.4,\ 18.4,\ 19.5,\ 20.7,\ 21.9,\ 23.2,\ 24.6,\ 26.1,\ 27.6,\ 29.3,\ 31.0)\ \text{cm}$$

from the mouth piece. It would be quite impractical with so many closely placed holes. In addition, ten fingers is not sufficient to cover them all simultaneously. Instead one uses *small* holes, which results in an effect in between completely covering or opening the hole. The holes can be placed more sparsely and fewer are needed, since more frequencies can be obtained by different combinations of open and closed holes.

Now to the actual **assignment**: Start by non-dimensionalizing (5) through the variable changes $\tau = Lt/c_0$ and $\xi = Lx$. Let $u(t,x) := p(Lt/c_0, Lx)$. Show that $u(t,x)$ satisfies

$$u_{tt} - u_{xx} + d(x)u = g(t,x), \qquad u(t,0) = u(t,1) = 0,$$

where $d(x)$ och $g(t,x)$ are somewhat modified as compared to $D(\xi)$ och $G(\tau,\xi)$.

Use the program you wrote in *a*), but modify it such that $d(x)u$ and the forcing function $g(t,x)$ are included. The Leap-Frog method becomes

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = \lambda^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n) - (\Delta t)^2 d(x_j)u_j^n + (\Delta t)^2 g(t_n, x_j).$$

In analogy with (3) this can be seen as a numerical solver of the ODE

$$\mathbf{u}_{tt} = \mathbf{A}\mathbf{u} - \mathbf{D}\mathbf{u} + \mathbf{g}(t), \tag{6}$$

based on central differencing in time. Here, $\mathbf{D} = \mathrm{diag}(d(x_j))$, a diagonal matrix, och $\mathbf{g}(t) = \{g(t,x_j)\}$. Make sure that A och D are stored as sparse matrices in your implementation.

Let $u(t,x) \equiv 0$ for $t \leq 0$ (dvs $u_j^{-1} = u_j^0 \equiv 0$) and

$$g(t,x) = e^{-10t}\sin(\pi x).$$

First do a simulation with all finger holes closed, i.e. $d(x) \equiv 0$. Solve the problem with $N = 128$ for a long time, until $t = 20$, and sample in each time step the solution at $x = 0.25$. Save the sampled values in a long vector, so that you obtain an approximation of $u(t, 0.25)$ for $t \in [0, 20]$. Fourier transform the values in the long vector with `Matlabs` `fft` routine and plot the absolute value of the components with `semilogy`. Rescale the variables such that the x-axis in the plot corresponds to the frequencies in a flute of length $L = 0.33$ m, and display only frequencies up to 2000 Hz. Does the first spike in the plot correspond to the theoretical base (fundamental) frequency $c_0/2L \approx 523$ Hz?

Repeat the experiment with a variable $d(x)$, as given by

$$d(x) = s d_h\left(\frac{x - \hat{x}}{w}\right), \qquad d_h(z) = 1000\begin{cases} 1 + \cos(2\pi z), & |z| \leq 1/2, \\ 0, & |z| > 1/2. \end{cases}$$

This models a finger hole where $\hat{x}$ is the x-coordinate for the hole, w is the width of the hole and s a measure of the size of the hole. Choose $w = 0.05$ and try with different sizes of s and different positions $\hat{x}$. How does the base frequency change? Is it true that a big hole ($s \geq 1$) yields about the same effect as a shorter flute with the length $\hat{x}$? Include frequency diagrams according to above for some selected cases. Try to have several holes open, e.g. $d(x) = s_1 d_h((x - \hat{x}_1)/w) + s_2 d_h((x - \hat{x}_2)/w)$ for some choices of s_1, s_2, $\hat{x}_1$, $\hat{x}_2$.

c) There is a more direct way to compute the basic frequency of the system. A standing wave with frequency f and amplitude $v(x)$ can be written $U(t,x) = v(x)\exp(i2\pi ft)$. It satisfies the same wave equation, but without the forcing function $G(t,x)$, i.e.

$$U_{tt} - U_{xx} + d(x)U = 0, \qquad U(t,0) = U(t,1) = 0,$$

Since $U_{tt} = -(2\pi f)^2 U$ we obtain an equation for v,

$$-v_{xx} + d(x)v = (2\pi f)^2 v, \qquad v(0) = v(1) = 0,$$

i.e. $(2\pi f)^2$ is an eigenvalue and $v(x)$ an eigen function to the operator $-\partial_{xx} + d(x)$.

The numerical approximation to this operator is simply the matrix $-\mathbf{A} + \mathbf{D}$ in (6). Go through the cases that you have reported in b). Compute the eigenvalues and eigenvectors to $-\mathbf{A} + \mathbf{D}$ for these $d(x)$. State the frequency that is given by the smallest eigenvalue - does it agree with the spikes in the frequency diagram? Plot the corresponding eigenvector (eigen function).

Optional excercise: Place seven holes with different coordinates $\hat{x}_k$ and sizes s_k, w_k, that is

$$d(x) = \sum_{k=1}^{7} s_k d_h \left(\frac{x - \hat{x}_k}{w_k} \right).$$

Try to select $\hat{x}_k$, s_k och w_k such that one can play the full scale mentioned above, $(f_0, \ldots, f_{12})$ through different combinations of open and closed holes. The top two holes on a real recorder are really two pairs of small holes next to each other. You can therefore allow for two different sizes s for these holes.