ROYAL INSTITUTE
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Linear Algebra, part 1

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Linear systems of equations

Let $\mathbf{A}$ be an $m \times n$ matrix (m rowns, n columns).
$(\mathbf{A})_{i,j} = a_{ij}$, i row index, j column index.
The matrix entries can be real ($a_{ij} \in \mathbb{R}$) or complex ($a_{ij} \in \mathbb{C}$).

Let $\mathbf{x}$ be a column vector of size $n \times 1$. $\mathbf{x} = (x_1, x_2, \ldots, \mathbf{x}_n)^T$.

Matrix vector multiplication: $\mathbf{b} = \mathbf{A}\mathbf{x}$,
where $\mathbf{b}$ is a column vector of size $m \times 1$. When $\mathbf{b}$ is given and $\mathbf{x}$ is unknown, want instead to solve (now assume $m = n$):

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

When does this system have a solution? When is it unique?

[WORKSHEET]

Range, rank, nullspace, nullity

Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$, where each $\mathbf{a}_i$ is $m \times 1$.
Then $\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$.

Column space $V = R(\mathbf{A})$ (*range* of $\mathbf{A}$) spanned by the columns of $\mathbf{A}$.
$rank(\mathbf{A}) = dim(V) =$ number of linearly independent columns.

If $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}$ is in the *nullspace* of $\mathbf{A}$.
$ker(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R} : \mathbf{Ax} = \mathbf{0}\}$.
The dimension of the nullspace: $nullity(\mathbf{A}) = dim(ker(\mathbf{A}))$.
If $\mathbf{A}$ is $m \times n$, we have that

$$rank(\mathbf{A}) + nullity(\mathbf{A}) = n.$$

Questions

Let $\mathbf{A}$ be $n \times n$.

1. What do we call A when $nullity(\mathbf{A}) > 0$?
2. When does $\mathbf{Ax} = \mathbf{b}$ have a unique solution?
3. When can we pick vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ such that $\mathbf{Ax} = \mathbf{b}_1$ have multiple solutions and $\mathbf{Ax} = \mathbf{b}_2$ have no solutions? What can we say about $\mathbf{b}_1$ and $\mathbf{b}_2$?

Large branch of numerical linear algebra

Solve linear system
$\mathbf{Ax} = \mathbf{b}$, $\mathbf{A}$ is $n \times n$
where n often is a large number.

Where do such systems come from?

- Discretization of differential equations by different numerical methods. (Applications to fluid mechanics, electromagnetics, quantum physics, biology, option pricing...)
- Network models and graphs (Electric circuits, mechanical trusses, hydraulic systems).

Example [NOTES]

Solution method: Gaussian elimination

Solution by Gaussian elimination.
(In Matlab: `x=A\b`)
Boxed material below taken from
Strang, Computational Science and Engineering, 2007.

$$\mathbf{Ku}=\mathbf{f} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \quad \text{is} \quad \begin{aligned} 2u_1 - u_2 \qquad &= 4 \\ -u_1 + 2u_2 - u_3 &= 0 \\ - u_2 + 2u_3 &= 0 \end{aligned}$$

The first step is to eliminate u_1 from the second equation. **Multiply equation 1 by $\frac{1}{2}$ and add to equation 2.** The new matrix has a zero in the 2, 1 position—where u_1 is eliminated. I have circled the **first two pivots**:

$$\begin{bmatrix} \textcircled{2} & -1 & 0 \\ 0 & \textcircled{\frac{3}{2}} & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \frac{1}{2} f_1 \\ f_3 \end{bmatrix} \quad \text{is} \quad \begin{aligned} 2u_1 - u_2 \qquad &= 4 \\ \tfrac{3}{2} u_2 - u_3 &= 2 \\ - u_2 + 2u_3 &= 0 \end{aligned}$$

Now, multiply Eq 2 by 2/3 and add to Eq 3.

Gaussian elimination, example continued

This yields:

$$\begin{bmatrix} \textcircled{2} & -1 & 0 \\ 0 & \textcircled{\frac{3}{2}} & -1 \\ 0 & 0 & \textcircled{\frac{4}{3}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \frac{1}{2} f_1 \\ f_3 + \frac{2}{3} f_2 + \frac{1}{3} f_1 \end{bmatrix} \quad \text{is} \quad \begin{aligned} 2u_1 - u_2 \qquad &= 4 \\ \tfrac{3}{2} u_2 - u_3 &= 2 \\ \tfrac{4}{3} u_3 &= \tfrac{4}{3} \end{aligned} \qquad (1)$$

Upper triangular matrix **U**.
Forward elimination is complete.

Solution by backsubstitution. Last equation determines u_3. Then the second determines u_2. With u_3 and u_2 known, easy the find u_1 using the first equation.

LU factorization

Note about the multipliers: When we know the pivot in row j, and we know the entry to be eliminated in row i, the multiplier ℓ_{ij} is their ratio:

$$\textbf{Multiplier} \quad \ell_{ij} = \frac{\textbf{entry to eliminate}}{\textbf{pivot}} \quad \frac{(\textit{in row } i)}{(\textit{in row } j)} \tag{4}$$

The convention is to **subtract** (*not add*) ℓ_{ij} times one equation from another equation.

Put the multipliers ℓ_{21},ℓ_{31}, ℓ_{32} etc. into a lower triangular matrix **L**.
This yields the LU factorization of **K**:

$$\boldsymbol{K = LU} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}. \tag{5}$$

Example 1

Example 1 Add -1's in the corners to get the circulant C. The first pivot is $d_1 = 2$ with multipliers $\ell_{21} = \ell_{31} = -\frac{1}{2}$. The second pivot is $d_2 = \frac{3}{2}$. But there is no third pivot:

$$C = \begin{bmatrix} \textcircled{2} & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{2} & -1 & -1 \\ 0 & \textcircled{\frac{3}{2}} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{2} & -1 & -1 \\ 0 & \textcircled{\frac{3}{2}} & -\frac{3}{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = U.$$

In the language of linear algebra, the rows of C are **linearly dependent**. Elimination found a combination of those rows (it was their sum) that produced the last row of all zeros in U. With only two pivots, C is **singular**.

A full set of pivots can not be found. **C** does not have full rank.
C is singular.

Example 2

Example 2 Suppose a zero appears in the second pivot position but there is a nonzero below it. Then a row exchange produces the second pivot and elimination can continue. This example is **not singular**, even with the zero appearing in the $2,2$ position:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ leads to } \begin{bmatrix} 1 & 1 & 0 \\ 0 & \mathbf{0} & 1 \\ 0 & \mathbf{1} & 1 \end{bmatrix}. \text{ Exchange rows to } U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exchange rows on the right side of the equations too! The pivots become all ones, and elimination succeeds. The original matrix is invertible but not positive definite. (Its determinant is *minus* the product of pivots, so -1, because of the row exchange.)

Permutation matrix **P** swaps the rows. Now, **PA** = **LU**.
L: Lower triangular matrix with 1s on the diagonal.
U: Upper triangular matrix.

A full set of pivots can be found, if the rows are swapped. The matrix does have full rank. The matrix is not singular.

Factorization and determinants

A matrix **A** is non-singular *if and only if* it admits a factorization **PA** = **LU**, where **P** is a row-reordering matrix.
(**P** = **I**, the identity matrix if no reordering necessary).

Computation of determinants:
$\det(\mathbf{PA}) = \det(\mathbf{P}) \cdot \det(\mathbf{A}) = \pm 1 \cdot \det(\mathbf{A})$.
$\det(\mathbf{LU}) = \det(\mathbf{L}) \cdot \det(\mathbf{U}) = \det(\mathbf{U})$ = product of all pivots.
Hence, if there are n non-zero pivots, then $\det(\mathbf{A}) \neq 0$.

Theorem:
A is non-singular *if and only if* $\det(\mathbf{A}) \neq 0$.

Symmetric matrices

Assume $\mathbf{A}$ symmetric, such that $\mathbf{A} = \mathbf{A}^T$.
If there is a factorization $\mathbf{A} = \mathbf{LU}$, it can also be written
$\mathbf{A} = \mathbf{LU} = \mathbf{LDL}^T$, where $\mathbf{D} = diag(\mathbf{U})$.

If all pivots are positive, we can write
$\mathbf{A} = \mathbf{L}_1\mathbf{L}_1^T$, where $\mathbf{L}_1 = diag(\sqrt{u_{ii}})\mathbf{L}$.
which is the Cholesky factorization.

If the pivots are all positive, the matrix is SPD - symmetric and positive definite.

Definition: A matrix $\mathbf{A}$ is SPD if it is symmetric and $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x}$.

Example: The so called normal equations: $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. Here, $\mathbf{A}$ is $m \times n$. If the columns of $\mathbf{A}$ are linearly independent, then $\mathbf{A}^T\mathbf{A}$ is SPD.

Show it! [Notes]