



Linear Algebra, part 2

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Mathematical Models, Analysis and Simulation
Fall semester, 2010

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Main problem of linear algebra 2:

Given an $n \times n$ matrix \mathbf{A} , find eigenvector(s) and eigenvalue(s) \mathbf{x} and λ such that

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

Rewrite the equation as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

This equation can only have a non-trivial solution if the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is singular.

The number λ is an eigenvalue of A if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

$\det(\mathbf{A} - \lambda\mathbf{I})$ is a polynomial in λ of degree n .

Will have n roots, $\lambda_1, \dots, \lambda_n$.

Can have multiplicity larger than one, and can have complex eigenvalues, even if A is real.

Can compute eigenvalues by hand using this approach for small systems. Other approaches needed for computer algorithms for large systems. Not the focus here.

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Facts about eigenvalues

The product of the n eigenvalues equals the determinant of A :
 $\det(A) = (\lambda_1) \dots (\lambda_n)$.

The sum of the eigenvalues equals the trace (sum of diagonal entries) of \mathbf{A} : $\lambda_1 + \lambda_2 + \dots + \lambda_n = a_{11} + a_{22} + \dots + a_{nn}$.

If \mathbf{A} is triangular, then its eigenvalues lie along the main diagonal.

The eigenvalues of A^2 are $\lambda_1^2, \dots, \lambda_n^2$.

The eigenvalues of A^{-1} are $1/\lambda_1, \dots, 1/\lambda_n$.

Eigenvectors of A are also eigenvectors of A^2 and A^{-1} (and any function of A).

Eigenvalues of $\mathbf{A} + \mathbf{B}$ and \mathbf{AB} are in general not known from eigenvalues of \mathbf{A} and \mathbf{B} , except for the special case when \mathbf{A} and \mathbf{B} commute, i.e. when $\mathbf{AB} = \mathbf{BA}$.

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Linear differential equations with constant coefficients

The simple equation

$$\frac{dy}{dt} = ay$$

has the general solution $y(t) = Ce^{at}$. (Initial condition det. C).

Consider a system (\mathbf{u} is $n \times 1$ vector, \mathbf{A} is $n \times n$):

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}.$$

Let \mathbf{x}_i be an eigenvector of \mathbf{A} with corresponding eigenvalue λ_i , and define $\mathbf{u}_i = e^{\lambda_i t} \mathbf{x}_i$. We then have

$$\mathbf{A}\mathbf{u}_i = \mathbf{A}(e^{\lambda_i t} \mathbf{x}_i) = e^{\lambda_i t} \mathbf{A}\mathbf{x}_i = e^{\lambda_i t} \lambda_i \mathbf{x}_i = \lambda_i \mathbf{u}_i$$

which is equal to $\frac{d\mathbf{u}_i}{dt} = \frac{d}{dt}(e^{\lambda_i t} \mathbf{x}_i) = \lambda_i e^{\lambda_i t} \mathbf{x}_i = \lambda_i \mathbf{u}_i$.

The general solution is

$$\mathbf{u}(t) = C_1 e^{\lambda_1 t} \mathbf{x}_1 + \dots + C_n e^{\lambda_n t} \mathbf{x}_n$$

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Growth and decay

For the scalar equation, solution $u(0)e^{at}$ decays if $a < 0$, grows if $a > 0$.
If a is complex, the real part of a determines the growth or decay.

For the system, the λ_i s determine which modes that will grow and which that will decay.

Example 1 - A 2×2 system with real eigenvalues.

Example 2 - Rigid body rotation: complex eigenvalues.

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Diagonalization of a matrix

Suppose that the $n \times n$ matrix \mathbf{A} has n linearly independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. Form a matrix \mathbf{S} , whose columns are the eigenvectors of \mathbf{A} . Then, $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}$ is diagonal. The diagonal entries of $\mathbf{\Lambda}$ are the eigenvalues $\lambda_1, \dots, \lambda_n$.

If \mathbf{A} has no repeated eigenvalues, i.e. all λ_i s are distinct, then \mathbf{A} has n linearly independent eigenvectors, and \mathbf{A} is diagonalizable.

If one or more eigenvalues of \mathbf{A} have multiplicity larger than one \mathbf{A} might have or might not have a full set of linearly independent eigenvectors.

If \mathbf{A} is symmetric, all eigenvalues are real and it has a full set of *orthonormal* eigenvectors. Denote by \mathbf{Q} the orthonormal matrix whose columns are eigenvectors of \mathbf{A} . (Defined on p 54 in Strang).
We have that $\mathbf{Q}^{-1} = \mathbf{Q}^T$ and $\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{\Lambda}$.

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Vector and Matrix norms, Quadratic forms

Euclidean norm of \mathbf{x} : $\|\mathbf{x}\|_2 = (\sum_{k=1}^n |x_k|^2)^{1/2} = \sqrt{\mathbf{x}^T \mathbf{x}}$.

2-norm of matrix defined as : $\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2} = \max \sqrt{\frac{\mathbf{x}^T \mathbf{Ax}}{\mathbf{x}^T \mathbf{x}}}$.

Consider the Rayleigh quotient: $R_{\mathbf{K}}(\mathbf{x}) = (\mathbf{x}^T \mathbf{Kx})/(\mathbf{x}^T \mathbf{x})$.

Differentiating $R_{\mathbf{K}}(\mathbf{x})$ with respect to x_k (DO IT!), can show that $\frac{\partial R}{\partial x_k} = 0$, $k = 1, \dots, n$ if and only if $\mathbf{Kx} = R_{\mathbf{K}}(\mathbf{x})\mathbf{x}$.

Theorem: If $\mathbf{Kx}^* = \lambda \mathbf{x}^*$, then \mathbf{x}^* is a stationary point of $R_{\mathbf{K}}(\mathbf{x})$ and $\lambda = R_{\mathbf{K}}(\mathbf{x}^*)$.

Hence, the Rayleigh quotient is maximized by the largest eigenvalue of \mathbf{K} .

Denote the largest eigenvalue of $\mathbf{A}^T \mathbf{A}$ by λ_M . We get $\|\mathbf{A}\|_2 = \sqrt{\lambda_M}$. This is well defined, since $\mathbf{A}^T \mathbf{A}$ is SPD and all eigenvalues are positive.

Navigation icons: back, forward, search, etc.

Positive definite matrices and minimum principles

A symmetric matrix \mathbf{A} is positive definite (SPD, introduced last time) if $\mathbf{x}^T \mathbf{Ax} > 0$ for all non-zeros vectors \mathbf{x} .

We know that a matrix is SPD if it is symmetric and all pivots are positive, or equivalently, that all eigenvalues are positive.

The solution \mathbf{x} to $\mathbf{Ax} = \mathbf{b}$ can also be viewed as a solution to the following minimization problem:

If \mathbf{A} is positive definite, then the quadratic $P(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{Ax} - \mathbf{x}^T \mathbf{b}$ is minimized at the point where $\mathbf{Ax} = \mathbf{b}$.
The minimum value is $P(\mathbf{A}^{-1} \mathbf{b}) = -\frac{1}{2} \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b}$.

Proof: [NOTES]

Example by calculus [NOTES]

Navigation icons: back, forward, search, etc.

Least squares solution

Consider again $\mathbf{Ax} = \mathbf{b}$, where \mathbf{A} is $m \times n$ with $m > n$.
(\mathbf{x} is $n \times 1$, \mathbf{b} is $m \times 1$).

Example: Find the line $y = c + dt$ that passes through four given points $(t_1, y_1), \dots, (t_4, y_4)$. If the four points all fall on a straight line, this overdetermined system has a solution. Otherwise, we want to find the straight line that "best" fit the data points.

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Normal equations

The vector \mathbf{x} that minimizes $\|\mathbf{Ax} - \mathbf{b}\|^2$ is the solution to the *normal equations*

$$\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}.$$

This vector $\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$ is the *least squares solution* to $\mathbf{Ax} = \mathbf{b}$.

Proof: $\|\mathbf{Ax} - \mathbf{b}\|^2 = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) = ((\mathbf{Ax})^T - \mathbf{b}^T)(\mathbf{Ax} - \mathbf{b}) = \mathbf{x}^T \mathbf{A}^T \mathbf{Ax} - \mathbf{x}^T \mathbf{A}^T \mathbf{b} - \mathbf{b}^T \mathbf{Ax} + \mathbf{b}^T \mathbf{b}.$

$\mathbf{b}^T \mathbf{Ax}$ is a scalar, and can be transposed: $\mathbf{b}^T \mathbf{Ax} = (\mathbf{b}^T \mathbf{Ax})^T = \mathbf{x}^T \mathbf{A}^T \mathbf{b}.$
The term $\mathbf{b}^T \mathbf{b}$ is constant, and does not affect the minimization. So, the form to minimize (scaled by a factor of 2, which is arbitrary) is

$$P(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{A}^T \mathbf{Ax} - \mathbf{x}^T \mathbf{A}^T \mathbf{b}.$$

According to earlier thm, this quadratic form is minimized for \mathbf{x} which is a solution to $\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}.$

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