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Structures in equilibrium. Minimizing with constraints.

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A system in equilibrium

From earlier: Consider a system of springs and masses in equilibrium.

To obtain our system of equations, we applied three equations (Strang, sec. 2.1):

1) The forces should be in equilibrium. $\mathbf{f} = \mathbf{A}^T\mathbf{w}$
2) Hooke's law for springs. $\mathbf{w} = \mathbf{C}\mathbf{e}$.
3) Relation between elongation of springs and displacements of masses $\mathbf{e} = \mathbf{A}\mathbf{u}$.

($\mathbf{u}$ displacements, $\mathbf{w}$ tension in the springs (internal forces), $\mathbf{e}$ elongation of the springs, $\mathbf{f}$ external forces on the masses.)

The system is on the form: $\begin{bmatrix} -\mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$, where $\mathbf{C}$ is a diagonal matrix with positive entries on the diagonal.

This yields $\mathbf{A}^T\mathbf{C}\mathbf{A}\mathbf{x} = \mathbf{f} + \mathbf{A}^T\mathbf{C}\mathbf{b}$.
"Stiffness matrix" $\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A}$ is symmetric positive definite if the columns of $\mathbf{A}$ are linearly independent.

Incidence matrix A

A is the so called "incidence matrix" for the graph. Much used when we considered electric circuits. (Strang, Section 2.3).

Now, we will consider trusses - 2D structures of elastic bars joined at pin joints, where the bars can turn freely. (Section 2.4 of Strang).
Under the assumption of small deformations, and a linearization of the elongation equation, a system on the same form as before is obtained.
However, **A** will be different.

Stable and unstable trusses.

Assume that we have $\mathbf{e} = \mathbf{Au}$, where $\mathbf{u}$ is a vector of displacements, and $\mathbf{e}$ gives the stretching (elongation) of the bars.
(To have this relation, must linearize as we will see...)

- Stable truss The columns of **A** are linearly **independent**.
  1. The only solution to $\mathbf{Au} = \mathbf{0}$ is $\mathbf{u} = \mathbf{0}$.
  2. The force balance equation $\mathbf{A}^T\mathbf{w} = \mathbf{f}$ can be solved for every $\mathbf{f}$.
- Unstable truss The columns of **A** are linearly **dependent**.
  1. $\mathbf{Au} = \mathbf{0}$ has a non-zero solution. We can have displacements with no stretching.
  2. The force balance equation $\mathbf{A}^T\mathbf{w} = \mathbf{f}$ is not solvable for every $\mathbf{f}$, some forces cannot be balanced.

Two types of unstable trusses:

- *Rigid motion*: The truss translates and/or rotates as a whole.
- *Mechanism*: The truss deforms. Change of shape without any stretching.

Linearized relation displacements - stretching

- Let $\bar{U}_i = (X_i, Y_i)$, $i = 1, \ldots, N$ be the coordinates of the nodes without loading.
- Let $\bar{U}_i + \bar{u}_i = (X_i, Y_i) + (x_i, y_i)$, $i = 1, \ldots, N$ be the coordinates of the nodes with loading.
- Let the vector $\mathbf{u}$ be $[x_1, \ldots, x_N, y_1, \ldots, y_N]^T$.
- Then we have that

  $$\mathbf{e} = \mathbf{A}\mathbf{u}$$

  is the *linearized* relation between displacements and stretching. $\mathbf{e} = [e_1, \ldots, e_m]$ are the elongations of the m bars.

[NOTES]

Incidence matrix A for trusses.

- Denote by θ_{ij} the angle that a bar from node i to j makes with the x-axis.
- Let $\bar{\mathbf{A}}$ be the edge-node incidence matrix as earlier in Strang for the electric circuits.
- Replace the ± 1s by $\pm \cos \theta_{ij}$:s in $\bar{\mathbf{A}}$ to produce the $\mathbf{A}_{cos}$ matrix. Analogously for $\mathbf{A}_{sin}$.
- Define $\mathbf{A} = [\mathbf{A}_{cos} \ \mathbf{A}_{sin}]$.
- Assume N nodes that are not fixed, and m edges. Then $\mathbf{A}_{cos}$ and $\mathbf{A}_{sin}$ are $m \times N$, i.e. $\mathbf{A}$ is $m \times 2N$.
- Relation between displacements and elongation (stretching):

  $$\mathbf{e} = \mathbf{A}\mathbf{u},$$

  where $\mathbf{u} = [x_1, \ldots, x_N, y_1, \ldots, y_N]^T$, i.e. of size $2N \times 1$, and $\mathbf{e} = [e_1, \ldots, e_m]$ (size $m \times 1$) holds the elongations of the m bars.

NOTE: I have suggested one sorting of the vectors, can choose to do it differently. The sorting that is used is reflected in how the incidence matrix is defined.

Forces in equilibrium.

- Let $\mathbf{w} = [w_1, \ldots, w_m]$ be the internal forces in the m bars (along the bar, positive sign direction given by graph) .
- Define $\mathbf{f} = [f_1^x, \ldots, f_N^x, f_1^y, \ldots, f_N^y]^T$, where (f_i^x, f_i^y) is the externally applied force in node i.
- Then, we can again write the condition that forces should be in equilibrium on the form:

$$\mathbf{A}^T\mathbf{w} = \mathbf{f}.$$

System of equations for trusses

Applying also Hooke's law for the bars (elastic contant for each bar), we have our "usual" equations:

1) The forces should be in equilibrium. $\mathbf{f} = \mathbf{A}^T\mathbf{w}$
2) Hooke's law for the bars. $\mathbf{w} = \mathbf{Ce}$.
3) Linearized relation between elongation of bars and displacements of nodes $\mathbf{e} = \mathbf{Au}$.

($\mathbf{u}$ displacements, $\mathbf{w}$ tension in the springs (internal forces), $\mathbf{e}$ elongation of the springs, $\mathbf{f}$ external forces on the masses.)

The system is on the form: $\begin{bmatrix} -\mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$,
where $\mathbf{C}$ is a diagonal matrix with positive entries on the diagonal.

System of equations for trusses, contd.

Using $\mathbf{A} = [\mathbf{A}_{cos}\ \mathbf{A}_{sin}]$, the big matrix reads $\begin{bmatrix} -\mathbf{C}^{-1} & \mathbf{A}_{cos} & \mathbf{A}_{sin} \\ \mathbf{A}_{cos}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{sin}^T & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Eliminating $\mathbf{w}$, we obtain the system

$$\mathbf{Bu} = \mathbf{f},$$

where $\mathbf{B} = \begin{bmatrix} \mathbf{A}_{cos}^T\mathbf{C}\mathbf{A}_{cos} & \mathbf{A}_{cos}^T\mathbf{C}\mathbf{A}_{sin} \\ \mathbf{A}_{sin}^T\mathbf{C}\mathbf{A}_{cos} & \mathbf{A}_{sin}^T\mathbf{C}\mathbf{A}_{sin} \end{bmatrix}$.

Constrained optimization

Section 2.2 of Strang.
[NOTES ON 2D PROBLEM]
Consider the following problem:

$$\begin{aligned} &\text{Minimize } f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n \\ &\text{subject to } g(\mathbf{x}) = C. \end{aligned}$$

Introduce the so called Lagrange function defined by

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda(g(\mathbf{x}) - C)$$

where the scalar λ is called a *Lagrange multiplier*. (The λ term may be added or subtracted).
If $\mathbf{x}$ is a minimum for the original constrained problem, then there exists a λ s.t. $(\mathbf{x}, \lambda)$ is a stationary point for L.
Stationary point:

$$\begin{cases} \frac{\partial L}{\partial x_i} = 0 & i = 1, \dots, m \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases}$$

$\frac{\partial L}{\partial \lambda} = 0$ gives back the constraint.
[WORKSHEET]

Constrained optimization - Example with multiple constraints

We can also have multiple constraints, simply add them all.
Example: To find equilibrium configuration of a system, minimize the energy with the constraint that the forces are in balance.

- Assume m springs.
- Elongations: $\mathbf{e} = [e_1, \ldots, e_m]^T$, internal forces: $\mathbf{w} = [w_1, \ldots, w_m]^T$.

Hooke's law: $w_i = c_i e_i$, or $e_i = w_i / c_i$.

$$\begin{aligned} E(\mathbf{w}) &= \frac{1}{2} c_1 e_1^2 + \ldots + \frac{1}{2} c_m e_m^2 = \frac{1}{2} \frac{1}{c_1} w_1^2 + \ldots + \frac{1}{2} \frac{1}{c_m} w_m^2 \\ &= \frac{1}{2} \mathbf{w}^T \mathbf{C}^{-1} \mathbf{w} \quad \mathbf{C}^{-1} \text{diagonal matrix with } 1/c_i \text{ on the diagonal.} \end{aligned}$$

Force balance: $\mathbf{A}^T \mathbf{w} = \mathbf{f}$ at n nodes.

Want to minimize the energy $E(\mathbf{w})$ subject to the constraint $\mathbf{A}^T \mathbf{w} = \mathbf{f}$.

Constrained optimization - Example, continued

Minimize the energy $E(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{C}^{-1} \mathbf{w}$
subject to the constraint $\mathbf{A}^T \mathbf{w} = \mathbf{f}$.

Introduce Lagrange multipliers λ_i, $i = 1, \ldots, n$; $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$.
Define the Lagrange function:

$$L(\mathbf{w}, \boldsymbol{\lambda}) = \frac{1}{2} \mathbf{w}^T \mathbf{C}^{-1} \mathbf{w} - \boldsymbol{\lambda}^T (\mathbf{A}^T \mathbf{w} - \mathbf{f})$$

Differentiate, set all partial derivatives to zero:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{w}} &= \mathbf{C}^{-1} \mathbf{w} - \mathbf{A} \boldsymbol{\lambda} = 0 \\ \frac{\partial L}{\partial \boldsymbol{\lambda}} &= -\mathbf{A}^T \mathbf{w} + \mathbf{f} \ = 0. \end{aligned}$$

Hence we obtain,

$$\mathbf{w} = \mathbf{C} \mathbf{A} \boldsymbol{\lambda}, \quad \mathbf{A}^T \mathbf{w} = \mathbf{f}.$$

Same equations as obtained by graph theory earlier, with $\boldsymbol{\lambda} = \mathbf{u}$.
[NOTES for details.]